# IMS

# MATHS

# BOOK-15

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# MATHEMATICS

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## Differential Equations

**Differential eqn:** An equation involving derivatives of a dependent variable w.r.t one or more independent variables, is called a differential eqn.

**Ex:** (1) $\frac{dy}{dx} = x\log x$

(2) $\frac{d^2y}{dx^2} + 3x\left(\frac{dy}{dx}\right)^2 - 5y = \log x$

(3) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 12y = 5e^x + \sin x + x^3$

(4) $\left(\frac{d^3y}{dx^3}\right)^{2003} + P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)\,y = S(x)$.

(5) $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

(6) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz$

**Note:** $\frac{dy}{dx} = y'$ (or) $y^{(1)}$ (or) $y_1$ ; $\frac{d^2y}{dx^2} = y''$ (or) $y^{(2)}$ (or) $y_2$

$\frac{d^3y}{dx^3} = y'''$ (or) $y^{(3)}$ (or) $y_3$ ............ $\frac{d^ny}{dx^n} = y^{(n)}$ (or) $y_n$

**Types of Differential equations:**

(i) **Ordinary Diff. eqn:** An eqn involving the derivatives of a dependent variable w.r.t a single independent variable, is called an ordinary diff. eqn.

The above examples (1), (2), (3), & (4) are ordinary diff. eqns.

(ii) **Partial Diff. eqn:** An equation involving the derivatives of a dependent variable w.r.t more than one independent

variable, is called a partial diff. eqn.

∴ The above examples (5) & (6) are partial diff. eqns.

Order of a Diff. eqn: The order of the highest order derivative involving in a differential eqn is called the order of the diff. eqn.

Ex: (1) $\frac{d^2y}{dx^2} + 4y = e^x$. is of 2nd order.

(2) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 12y = 5e^x + \sin x + x^3$ is of second order.

(3) $\frac{d^2y}{dx^2} = k\left[1+\left(\frac{dy}{dx}\right)^3\right]^{5/3}$ is of 2nd order.

(4) $\log\left(\frac{dy}{dx}\right) = ax+by$ is of 1st order.

(5) $\sin\left(\frac{dy}{dx}\right) = x^{100}$
order = 1

(6) $\cos\left(\frac{dy}{dx}\right) = x^{100}$
order = 1

Note [1]. A differential eqn of order one is of the form $f(x, y, \frac{dy}{dx}) = 0$

[2]. A diff. eqn of order two is of the form $F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0$.

[3]. In general, diff. eqn of order 'n' is of the form $F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}) = 0$

Degree of a diff. eqn: The degree (i.e., power) of the highest order derivative involving in a diff. eqn, when the derivatives are made free from radicals (roots) and fractions, is called the degree of the diff. eqn.

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Ex: (1) $x\left(\frac{d^2y}{dx^2}\right)^3 + y^2\left(\frac{dy}{dx}\right)^4 + xy = 0$. is of order 2 and degree 3.

(2) $\frac{d^2y}{dx^2} = k\left[1+\left(\frac{dy}{dx}\right)^3\right]^{5/3}$ (radical form)

Cubing on both sides, we get,

$$\left(\frac{d^2y}{dx^2}\right)^3 = k^3\left(1+\left(\frac{dy}{dx}\right)^3\right)^5$$

order = 2
Degree = 3.

(3) $y\left(\frac{dy}{dx}\right) = \sqrt{x} + \frac{k}{dy/dx}$ (fractions form)

$$\Rightarrow y\left(\frac{dy}{dx}\right)^2 = \sqrt{x}\left(\frac{dy}{dx}\right) + k.$$

$\therefore$ order = 1
Degree = 2

(4) $y = x\frac{dy}{dx}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$\Rightarrow y^2 = x^2\left(\frac{dy}{dx}\right)^2\left(1+\left(\frac{dy}{dx}\right)^2\right)$$

$$\Rightarrow y^2 = x^2\left(\frac{dy}{dx}\right)^2 + x^2\left(\frac{dy}{dx}\right)^4.$$

Order = 1
Degree = 4

(5) $\frac{d^3y}{dx^3} = \sqrt{1+\left(\frac{dy}{dx}\right)^5}$

$$\Rightarrow \left(\frac{d^3y}{dx^3}\right)^2 = 1+\left(\frac{dy}{dx}\right)^5$$

Order = 3
degree = 2

(6) $\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$ fractions form

$$\Rightarrow \rho\left(\frac{d^2y}{dx^2}\right) = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

$$\Rightarrow \rho^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

$\therefore$ Order $=2$

Degree $=2$

(7) $$\left(y'''\right)^{4/3} + \sin x\left(\frac{dy}{dx}\right) + xy = x$$

$$\Rightarrow \left(y'''\right)^{4/3} = x - \sin x\left(\frac{dy}{dx}\right) - xy$$

$$\Rightarrow \left(y'''\right)^4 = \left(x - \sin x\left(\frac{dy}{dx}\right) - xy\right)^3$$

order $=3$

Degree $=4$

(8) $$\left(y'''\right)^{1/2} - 2\left(y'\right)^{1/4} + xy = 0$$

$$\Rightarrow \left(y'''\right)^{1/2} + xy = 2\left(y'\right)^{1/4}$$

$$\Rightarrow \left(\left(y'''\right)^{1/2} + xy\right)^4 = 2^4 y'$$

$$\Rightarrow \left[\left(\left(y'''\right)^{1/2} + xy\right)^2\right]^2 = 16y'$$

$$\Rightarrow \left[y''' + x^2y^2 + 2xy\left(y'''\right)^{1/2}\right]^2 = 16y'$$

$$\Rightarrow \left(y'''\right)^2 + x^4y^4 + 4x^2y^2y''' + 2x^2y^2y''' + 4x^3y^3\left(y'''\right)^{1/2} + 4xy\left(y'''\right)^{3/2} = 16y'$$

$$\Rightarrow 4xy\left(y'''\right)^{1/2}\left[y''' + x^2y^2\right] = \left[16y' - \left(y'''\right)^2 - x^4y^4 - 6x^2y^2y'''\right]$$

Squaring on both sides

$$16x^2y^2y'''\left(y''' + x^2y^2\right)^2 = \left(16y' - \left(y'''\right)^2 - x^4y^4 - 6x^2y^2y'''\right)^2$$

$$\Rightarrow 16x^2y^2y'''\left[\left(y'''\right)^2 + x^4y^4 + 2x^2y^2y'''\right] = \left(16y'\right)^2 + \left(y'''\right)^4 + \cdots\cdots$$

$\therefore$ order $=3$ & Degree $=4$

(9) $(y''')^{4/3} + (y')^{1/5} + y = 0$

Order $= 3$

Degree $= 60$

(10) $(y''')^{3/2} + (y''')^{2/3} = 0$

Order $= 3$

Degree $= \cancel{8}$

Note:

$\boxed{1}$ $y = \sin\left(\frac{dy}{dx}\right)$

Order $= 1$

Degree = not defined.

Because $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$

$= \frac{dy}{dx} - \frac{1}{3!}\left(\frac{dy}{dx}\right)^3 + \frac{1}{5!}\left(\frac{dy}{dx}\right)^5 - \frac{1}{7!}\left(\frac{dy}{dx}\right)^7 + \cdots\cdots$

Here $x = \frac{dy}{dx}$

Similarly $\cos\left(\frac{dy}{dx}\right)$; $\tan\left(\frac{dy}{dx}\right)$; $\cot\left(\frac{dy}{dx}\right)$, $\sec\left(\frac{dy}{dx}\right)$ and $\operatorname{cosec}\left(\frac{dy}{dx}\right)$ degrees do not exist (or not defined.

$\boxed{2}$. $y = x\left(\frac{dy}{dx}\right) + \sin\left(\frac{dy}{dx}\right)$

Order $= 1$

Degree = not defined.

(3) $\frac{d^2y}{dx^2} + 2e^{x\frac{dy}{dx}} - 3y = x$

$\Rightarrow 2e^{x\frac{dy}{dx}} = x + 3y - \frac{d^2y}{dx^2}$

$$\Rightarrow x\frac{dy}{dx} = \log\left[\frac{1}{2}\left(x+3y - \frac{d^2y}{dx^2}\right)\right]$$

$\therefore$ order $= 2$

Degree = not defined.

(4) $3x^2 \frac{d^3y}{dx^3} - \sin \frac{d^2y}{dx^2} - \cos(xy) = 0$

(5) $(y''')^{1/3} + xy'' = 2005$

$$\Rightarrow (y''')^{1/3} = -xy'' + 2005$$

$$\Rightarrow y''' = (2005 - xy'')^3$$

Order $= 3$

Degree $= 1$

(6) $\left[y'' - 4(y')^2\right]^{5/2} = ay''$

$$\left[y'' - 4(y')^2\right]^5 = a(y'')^2$$

order $= 2$

degree $= 5$

## Linear Differential Eqn:

A differential equation is said to be linear if (i) the dependent variable say 'y' and all its derivatives occur in the first degree only
(ii) no product of dependent variables (or) derivatives occur.

Ex: (1) $\frac{dy}{dx} = x + \sin x$

(2) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} = 0$

## Non-Linear diff. eqn:

A diff. eqn which is not linear is called a non-linear diff. eqn.

Ex (1) $\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t$.

(2) $y = \sqrt{x}\frac{dy}{dx} + \frac{k}{dy/dx}$

(3) $k\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$

(4) $\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial x^3}\right)^2$

Note: In general, a linear diff. eqn of $n^{th}$ order is of the form

$$\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots\cdots + P_{n-1}(x)\frac{dy}{dx} + P_n(x)y = Q(x).$$

Solution of a diff. eqn: Any relation between the dependent and independent variables which when we substituted in the diff. eqn reduces it to an identity is called a solution (or) integral (or) primitive of the diff. eqn.

Ex: $y = ce^{2x}$ is a sol$^n$ of the diff. eqn $y' = 2y$ — ①

because $y = ce^{2x} \Rightarrow y' = 2ce^{2x}$

$\therefore$ ① $\equiv$ $2ce^{2x} = 2[ce^{2x}]$ is an identity

General sol$^n$: The sol$^n$ of a diff. eqn in which the number of arbitrary constants is equal to the order of the diff. eqn.

Ex: $y = ce^{2x}$ is G.S. of the diff. eqn $y' = 2y$

$\therefore$ Arbitrary constants = order of the diff. eqn

Particular solution: A solution obtained by giving particular values to arbitrary constants in the general solution, is called a particular sol$^n$ (or) particular integral.

Ex: In the above example taking $c = 1$

$y = e^{2x}$ is a particular sol$^n$ of $y' = 2y$

Singular solution:

An eqn $\psi(x, y) = 0$ is called singular solution of the diff. eqn $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^ny}{dx^n}\right) = 0$

if (i) $\psi(x, y) = 0$ is a solution of the given diff. eqn.

(ii) $\psi(x, y) = 0$ doesnot contain arbitrary constants.

and (iii) $\psi(x, y) = 0$ is not obtained by giving particular values to arbitrary constants in the general sol$^n$.

Arbitrary constants: The complete sol$^n$ of a diff. eqn of the $n^{th}$ order contains exactly '$n$' arbitrary constants.

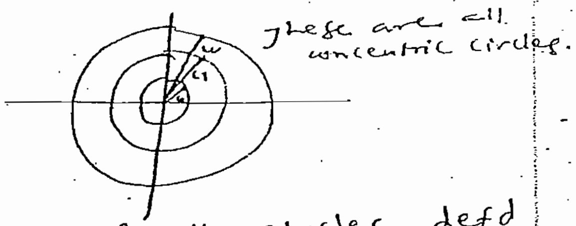
## Family of plane curves:

For each given set of real numbers $c_1, c_2, c_3, \ldots\ldots c_n$, the equation $\phi(x, y, c_1, c_2, \cdots c_n) = 0$ represents a curve in $xy$-plane.

For different sets of real values of $c_1, c_2, \ldots\ldots c_n$, the equation $\phi(x, y, c_1, c_2, \cdots c_n) = 0$ represents infinitely many curves.

The set of all these curves is called $n$-parameter family of curves. $c_1, c_2, \ldots\ldots c_n$ are called parameters of the family.

**Ex:-** (1) the set of concentric circles defined by $x^2 + y^2 = c$ is one parameter family if '$c$' takes all non-negative values.




(2) the set of all circles, defd by $(x - c_1)^2 + (y - c_2)^2 = c_3$ is a three-parameter family if $c_1, c_2$ take all real values and $c_3$ takes all non-negative real values.

## Formation of Diff. eqns:—

**working rule:**

To form the diff. eqn from a given eqn is $x$ and $y$, containing arbitrary constants:

Step 1: write down the given eqn.

Step 2:- Differentiate w.r.t '$x$' successively as many as the number of arbitrary constants.

Step 3: Eliminate the arbitrary constants from the given eqns of above two steps.

∴ The resulting is the required diff. eqn.

**Problems:**

(1) Find the diff. eqn of $y = Ae^{2x} + Be^{-3x}$; (A, B are arbitrary constants)

Sol:
$$y' = 2Ae^{2x} - 3Be^{-3x} \quad \text{—(1)}$$
$$y'' = 4Ae^{2x} + 9Be^{-3x} \quad \text{—(2)}$$

$$(1)+(2) \equiv y' + y'' = 6Ae^{2x} + 6Be^{-3x}$$
$$= 6(Ae^{2x} + Be^{-3x})$$
$$= 6y.$$

$$\therefore y' + y'' = 6y$$
$$\Rightarrow y'' + y' - 6y = 0$$
which is the required diff. eqn.

(or)

$$\begin{vmatrix} y & -e^{2x} & -e^{-3x} \\ y' & -2e^{2x} & 3e^{-3x} \\ y'' & -4e^{2x} & -9e^{-3x} \end{vmatrix} = 0 \Rightarrow e^{-2x} e^{-3x} \begin{vmatrix} y & -1 & -1 \\ y' & -2 & 3 \\ y'' & -4 & -9 \end{vmatrix} = 0$$

$\Rightarrow y(18+12)+1(-9y'-3y'')-1(-4y'+2y'')=0$

$\Rightarrow 30y-5y'-5y''=0$ $\quad (\because e^{2x}e^{-3x}\neq 0.)$

$\Rightarrow y''+y'-6y=0.$

(2) Find the diff eqn of the family of curves $y=a(x-a)^2$ ① where $a$ is an arbitrary constant.

Soln: Differentiate ① w.r.t $x$ we get,

$y'=2a(x-a)$ ②

Now $\frac{①}{②} \Rightarrow \frac{1}{2}(x-a)=\frac{y}{y'}$

$\Rightarrow 2y=y'(x-a)$

$\Rightarrow ay'=xy'-2y$

$\Rightarrow a=\frac{xy'-2y}{y'}$

$\therefore ① \equiv y=\left(\frac{xy'-2y}{y'}\right)\left[x-\left(\frac{xy'-2y}{y'}\right)\right]^2$

$=\left(\frac{xy'-2y}{y'}\right)\left(\frac{2y}{y'}\right)^2$

$y=\frac{(xy'-2y)4y^2}{(y')^3}$

$\Rightarrow (y')^3=(xy'-2y)4y.$

which is the required diff-eqn.

(3) Find the diff eqn of $y=Ae^{ax}+Be^{bx}$; $A,B$ are arbitrary constants.

Ans $\boxed{y''-(a+b)y'+aby=0}$

(4) Find the diff-eqn of $y=Ae^{-x}+Be^{2x}$; $A,B$ are arbitrary constants.

$y''-(-1+2)y'+(-1)(2)y=0$

$\Rightarrow y''-y'-2y=0$

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(5) Find ~~the diff. eqn~~ of $y = Ae^{ax} + Be^{bx} + Ce^{cx}$; (A, B, C)

$$\boxed{y''' - (a+b+c)y'' + (ab+bc+ca)y' - abcy = 0}$$

(6) Find the diff. eqn of $y = ae^{x} + be^{3x} + ce^{5x}$; (a, b, c)

$$y''' - (1+3+5)y'' + (1\cdot3 + 3\cdot5 + 5\cdot1)y' - (1\cdot3\cdot5)y = 0$$

$$\Rightarrow y''' - 9y'' + 23y' - 15y = 0$$

(7) Form the diff. eqn of $y = ae^{x} + be^{2x} + ce^{3x}$ where a, b, c arbitrary constants.

$$y''' - 6y'' + (2+6+3)y' - 6y = 0$$

$$\Rightarrow y''' - 6y'' + 11y' - 6y = 0$$

(8) $y = ae^{2x} + be^{-3x} + ce^{x}$; (a, b, c)

$$y''' - (2-3+1)y'' + (2(-3) + (-3)(1) + 1(2))y' - (2\cdot -3\cdot 1)y = 0$$

$$\Rightarrow y''' - 0y'' + (-6-3+2)y' + 6y = 0$$

$$\Rightarrow y''' - 7y' + 6y = 0$$

(9) Form the diff. eqn $y = ae^{3x} + be^{5x}$

Ans: $y'' - 8y' + 15y = 0$

(10) Form the diff. eqn of $y = e^{ax}(c_1 \sin bx + c_2 \cos bx)$ —(1)
($c_1, c_2$ are arbitrary constants)

$$y' = e^{ax}(c_1 b\cos bx - c_2 \sin bx) + ae^{ax}(c_1 \sin bx + c_2 \cos bx)$$

$$\Rightarrow y' = e^{ax}(c_1 b\cos bx - c_2 \sin bx) + ay \quad \text{(from (1))}$$

$$\Rightarrow y' - ay = e^{ax}(c_1 b\cos bx - c_2 b\sin bx) \quad \text{--- (2)}$$

$$\Rightarrow y'' - ay' = e^{ax}(-c_1 b^2 \sin bx - c_2 b^2 \cos bx) + a e^{ax}(c_1 b\cos bx - c_2 b \sin bx)$$

$$= -b^2 y + a(y' - ay) \quad (\text{by (1) \& (2)})$$

$$\Rightarrow y'' - ay' = -b^2 y + ay' - a^2 y$$

$$\Rightarrow \boxed{y'' - 2ay' + (a^2+b^2)y = 0}$$

(11) Form the diff. eqn of $y = e^{ax}(c_1 \sin bx - c_2 \cos bx)$ (1) where $c_1, c_2$ are arbitrary constants

$$y' = e^{ax}(c_1 b\cos bx + c_2 b \sin bx) + a e^{ax}(c_1 \sin bx - c_2 \cos bx)$$

$$\Rightarrow y' = e^{ax}(c_1 b\cos bx + c_2 b\sin bx) + ay \quad (\text{by (1)})$$

$$\Rightarrow y' - ay = e^{ax}(c_1 b\cos bx + c_2 b \sin bx) \quad \text{--- (2)}$$

$$\Rightarrow y'' - ay' = a e^{ax}(c_1 b\cos bx + c_2 b\sin bx) + e^{ax}(-c_1 b^2 \sin bx + c_2 b^2 \cos bx)$$

$$\Rightarrow y'' - ay' = a(y' - ay) - b^2 y$$

$$\Rightarrow \boxed{y'' - 2ay' + (a^2+b^2)y = 0}$$

(12) $y = e^{x}(c_1 \cos x + c_2 \sin x)$; $(c_1, c_2)$

$$y'' - 2(1)y' + (1^2+1^2)y = 0$$

$$\Rightarrow y'' - 2y' + 2y = 0$$

(13) $y = c_1 \cos 2x + c_2 \sin 2x$

$$y'' - 2(0)y' + (0^2+2^2)y = 0$$

$$\Rightarrow y'' - 4y = 0$$

(14) Find the diff. eqn of $Ax^2+By^2=1$ —(1) ; A, B are arbitrary constants.

Sol$^n$:
$$\begin{vmatrix} x^2 & y^2 & -1 \\ 2x & 2yy' & 0 \\ 2 & 2(yy''+(y')^2) & 0 \end{vmatrix} = 0$$

$$\Rightarrow -1\begin{vmatrix} 2x & 2yy' \\ 2 & 2(yy''+(y')^2) \end{vmatrix} = 0$$

$$\Rightarrow -1\left(4x(yy''+(y')^2)-4yy'\right)=0$$

$$\Rightarrow 4xyy''+4x(y')^2-4yy'=0$$

$$\Rightarrow x(yy''+(y')^2)-yy'=0$$

(OR)

$$2Ax+2Byy'=0$$

$$\Rightarrow \frac{yy'}{x}=-\frac{A}{B}$$

$$\Rightarrow \frac{x((y')^2+yy'')-yy'}{x^2}=0$$

$$\Rightarrow x((y')^2+yy'')-yy'=0$$

(15) Find the diff. eqn of the family of ellipses whose axes coincide with the axes of co-ordinates and centres at the origin.

i.e, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ —(1) ; a, b are arbitrary constants

$$\frac{2x}{a^2}+\frac{2yy'}{b^2}=0$$

$$\Rightarrow \frac{yy'}{x}=\frac{-b^2}{a^2} \quad \text{—(2)}$$

$$\frac{yy' - x\left((y')^2 + yy''\right)}{x^2} = 0$$

$$\Rightarrow x\left((y')^2 + yy''\right) - yy' = 0$$

(16) 1992 $xy = ae^{x} + be^{-x} + x^2$ ; $(a, b)$

$\Rightarrow xy - x^2 = ae^{x} + be^{-x}$ —— ①

$\Rightarrow xy' + y - 2x = ae^{x} - be^{-x}$

$\Rightarrow xy'' + y' + y' - 2 = ae^{x} + be^{-x}$

$\Rightarrow xy'' + 2y' - 2 = xy - x^2$ (from ①)

(or)

$(xy - x^2)'' - (0)(xy - x^2)' + (-1)(xy - x^2) = 0$

$\Rightarrow (xy' + y - 2x)' - xy + x^2 = 0$

$\Rightarrow xy'' + 2y' - 2 - xy + x^2 = 0$

(17) Form the diff. eqn of the family of circles, given by $x^2 + y^2 + 2ax + 2by + c = 0$ ; $a, b, c$ arbitrary constants —— ①

Sol$^n$

$2x + 2yy' + 2a + 2by' = 0$

$x + yy' + a + by' = 0$ —— ②

$1 + yy'' + (y')^2 + by'' = 0$

$\Rightarrow -b = \dfrac{1 + yy'' + (y')^2}{y''}$ —— ③

$$\frac{y''\left(yy''' + y'y'' + 2y'y''\right) - y'''\left(1 + yy'' + (y')^2\right)}{(y'')^2} = 0$$

$\Rightarrow y''\left(yy''' + 3y'y''\right) - y'''\left(1 + yy'' + (y')^2\right) = 0$

$\Rightarrow y'''\left(\cancel{yy''} - 1 - \cancel{yy''} - (y')^2\right) + 3y'(y'')^2 = 0$

$\Rightarrow y'''\left(1 + (y')^2\right) = 3y'(y'')^2$

(18) Find the diff. eqn of family of the curve $\frac{x^2}{a^2}+\frac{y^2}{a^2+\lambda}=1$ (9), where $\lambda$ is parameter.

Sol$^n$

$$\frac{x^2}{a^2}+\frac{y^2}{a^2+\lambda}=1 \quad \text{---} (1)$$

$$\frac{2x}{a^2}+\frac{2yy'}{a^2+\lambda}=0$$

$$\Rightarrow -\frac{1}{a^2+\lambda}=\frac{x^2}{a^2(yy')}$$

$$\Rightarrow \boxed{\frac{1}{a^2+\lambda^2}=-\frac{x^2}{a^2(yy')}}$$

$$\therefore (1) \equiv \frac{x^2}{a^2} - \frac{x^2y^2}{a^2yy'}=1$$

$$\Rightarrow \left(1-\frac{y}{y'}\right)=\frac{a^2}{x^2}$$

$$\Rightarrow y'-y=\frac{a^2}{x^2}y'$$

$$\Rightarrow \left(1-\frac{a^2}{x^2}\right)y'-y=0.$$

which is the required diff. equation.

[19] $y=ax^2+bx$ ; $(a,b)$ —(1)

Sol $y'=2ax+b$ —(2)

$y''=2a$ —(3)

Eliminating the arbitrary constants from (1), (2) and (3), we have

$$\begin{vmatrix} y & -x^2 & -x \\ y' & -2x & -1 \\ y'' & -2 & 0 \end{vmatrix}=0 \Rightarrow x^2y''+2xy'-2x^2y''-2y=0$$

$$\Rightarrow x^2y''-2xy'+2y=0.$$

[20] $y = ax + be^{x}$ ; $(a, b)$ —— ①

Sol $y' = a + be^{x}$ —— ②

$y'' = be^{x}$ —— ③

Now eliminating, a & b from ①, ② & ③, we get

$$\begin{vmatrix} y & -x & -e^{x} \\ y' & -1 & -e^{x} \\ y'' & 0 & -e^{x} \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} y & -x & -1 \\ y' & -1 & -1 \\ y'' & 0 & -1 \end{vmatrix} = 0 \quad (\because e^{+x} \neq 0)$$

$\Rightarrow y(1) + x(-y' + y'') - 1(y'') = 0$

$\Rightarrow (x-1)y'' - xy' + y = 0.$

HW

[21] $y = ax^{2} + be^{x}$ ; $(a, b)$.

[22] $x = A \sin t + B \cos t + t \sin t$ ; $(A, B)$ —— ①

Sol $x - t \sin t = A \sin t + B \cos t$ —— ②

$\Rightarrow x' - t \cos t - \sin t = A \cos t - B \sin t$ —— ③

$\Rightarrow x'' - \cos t + t \sin t - \cos t = -A \sin t - B \cos t$

$\Rightarrow x'' - 2\cos t + t \sin t = -(A \sin t + B \cos t)$

$\Rightarrow x'' - 2\cos t + t \sin t = x - t \sin t$ (by ②)

$\Rightarrow \boxed{x'' - 2\cos t + 2t \sin t = x}$

[23] $y = A \sin(mx + \alpha)$ —— ① $(\alpha, A)$

Sol $y' = [A \cos(mx + \alpha)]\, m$ —— ②

$y'' = -[A \sin(mx + \alpha)]\, m^{2}$

$\boxed{y'' = -y m^{2}}$ which is the reqd diff. eqn.

② $\equiv \quad \cdots h = \dfrac{[1+(y')^2]}{y''} y'$

$\therefore$ ① $\equiv \dfrac{[1+(y')^2]^2}{(y'')^2}(y')^2 + \dfrac{[1+(y')^2]^2}{(y'')^2} = r^2$

$\Rightarrow \dfrac{[1+(y')^2]^2}{(y'')^2}[(y')^2+1] = r^2$

$\Rightarrow \boxed{[1+(y')^2]^3 = r^2 (y'')^2}$

[28] $y = ae^x + be^{-x} + c\cos x + d\sin x$ $(a, b, c, d)$ — ①

Sol $y' = ae^x - be^{-x} - c\sin x + d\cos x$ — ②

$y'' = ae^x + be^{-x} - c\cos x - d\sin x$ — ③

$y''' = ae^x - be^{-x} + c\sin x - d\cos x$ — ④

$y^{iv} = ae^x + be^{-x} + c\cos x + d\sin x$ — ⑤

$\Rightarrow \boxed{y^{iv} = y}$ (from ①).

which is the required diff. eqn.

HW

[29] Find the diff. eqn of the family of straight lines $y = mx + \frac{a}{m}$ where 'm' is the parameter.

HW

[30] Find the diff. eqn of family of all circles of fixed radius 'r' and centres on y-axis i.e. $x^2 + (y-k)^2 = r^2$ (k is parameter, r is constant).

24 $x = A\cos(pt - \alpha)$ ; $(A, \alpha)$

Ans: $\boxed{x'' = -p^2 x}$

25 $y^2 = 4ax$ ; 'a' is parameter. —— ①

$2yy' = 4a$

$\therefore ① \equiv \boxed{y^2 = (2yy')\,x}$

26 $y = a\cos(x+3)$ ; a is parameter —— ①

$y' = -a\sin(x+3)$ —— ②

$\frac{①}{②} \equiv \frac{y}{y'} = -\cot(x+3)$

$\Rightarrow \frac{y'}{y} = -\tan(x+3)$

$\Rightarrow \boxed{y' + y\tan(x+3) = 0}$

27 Find the diff. equation of the family of circles

$(x-h)^2 + (y-k)^2 = r^2$ where h and k are parameters & r is fixed constant.

Sol. $(x-h)^2 + (y-k)^2 = r^2$ —— ①

$(x-h) + (y-k)y' = 0$ —— ②

$\Rightarrow 1 + (y-k)y'' + (y')^2 = 0$

$\Rightarrow \boxed{(y-k) = -\frac{[1+(y')^2]}{y''}}$

HW

31. Find the diff. eqn of all circles of fixed radius 'r' and centres on x-axis is $(x-h)^2+y^2=r^2$, h is parameter.

HW

32. The family of all circles touching x-axis at the origin is $x^2+y^2-2ay=0$.

HW

33. The family all circles touching y-axis at the origin is $x^2+y^2-2ax=0$. ('a' is parameter).

## * Solution of Differential equations of the first order and first degree:-

Defn: A diff. eqn of first order and first degree is an eqn of the form $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ (or) $M\,dx+N\,dy=0$ where M and N are functions of x & y.

The first order, first degree diff. eqns solving into four methods

(i) Variables seperable (ii) Homogeneous equations (iii) Exact eqns (iv) Linear equations.

### (i) Variables seperable:

If in an eqn, it is possible to get all the functions of x and dx to one side, and all the functions of y and dy to anotherside, then the variables are said to be seperable

HW

[31] Find the diff. eqn of all circles of fixed radius 'r' and centres on x-axis is :- $(x-h)^2+y^2=r^2$, h is parametre.

HW

[32] The family of all circles touching x-axis at the origin is $x^2+y^2-2ay=0$.

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The first order, first degree diff. eqns solving into four methods

(i) Variables seperable (ii) Homogeneous equations (iii) Exact eqns (iv) Linear equations.

(i) Variables seperable:

If in an eqn, it is possible to get all the functions of x and dx to one side, and all the functions of y and dy to anotherside, then the variables are said to be seperable

**Working rule :-**

**Step(1):** Consider the equation $\frac{dy}{dx} = XY$; where $X$ is a function of $x$ only and $Y$ is a function of $y$ only.

**Step(2):** $\frac{dy}{Y} = X\,dx$; i.e. the variables have been seperated.

**Step(3):** Integrating on both sides,

$$\int \frac{dy}{Y} = \int X\,dx + C,$$ where 'C' is an arbitrary constant.

**Note :-** (1) Never forget to add an arbitrary constant on one side (only). A solution without this arbitrary constant is wrong, for it is not a general solution.

(2). The nature of the arbitrary constant depends upon the nature of the problem.

**Problems.**

→ Solve the following diff. eqns.

[1] $\frac{dy}{dx} = e^{x+y} + x^2 e^y$ —— (1)

$\frac{dy}{dx} = e^x . e^y + x^2 e^y$

$\Rightarrow \frac{dy}{dx} = e^y (e^x + x^2)$

$\Rightarrow \frac{1}{e^y} dy = (e^x + x^2) dx$ i.e. the variables have been seperated

Now integrating on both sides, we get

$$\int \frac{dy}{e^y} = \int (e^x + x^2)\, dx + c$$

$$\Rightarrow -e^{-y} = e^x + \frac{x^3}{3} + c.$$

$$\Rightarrow \boxed{e^{-y} + e^x + x^3/3 + c = 0}$$ which is the reqd g.s.

[2] $\frac{dy}{dx} = e^{x-y} + e^{2\log x - y}$

$$\Rightarrow \frac{dy}{dx} = e^x \cdot e^{-y} + e^{2\log x} \cdot e^{-y}$$

$$\Rightarrow \frac{dy}{dx} = e^{-y}(e^x + x^2)$$

$$\Rightarrow e^y dy = (e^x + x^2)\, dx$$

Integrating on both sides, we get

$$\boxed{e^y = e^x + x^3/3 + c}$$ which is the reqd g.s.

[3] $y \frac{dy}{dx} = x e^{x^2 + y^2}$

$$y \frac{dy}{dx} = \frac{x e^{x^2}}{e^{-y^2}} \Rightarrow y e^{-y^2} dy = x e^{x^2}$$

integrating on both sides, we get

$$\int y e^{-y^2} dy = \int x e^{x^2} dx + c$$

$$\Rightarrow -\frac{e^{-y^2}}{2} = \frac{e^{x^2}}{2} + \frac{c}{2}$$ put $y^2 = t$, $y\,dy = \frac{dt}{2}$

$$\Rightarrow \boxed{e^{-y^2} + e^{x^2} + c = 0}$$

14 $\frac{dy}{dx} + \frac{1+y^2}{1+x^2} = 0$

$\Rightarrow \frac{dy}{dx} = -\left[\frac{1+y^2}{1+x^2}\right]$

$\Rightarrow \frac{dy}{1+y^2} + \frac{dx}{1+x^2} = 0$

$\Rightarrow \boxed{\tan^{-1}y + \tan^{-1}x = \tan^{-1}c}$

15 $y\,dx - x\,dy = xy\,dx$

$\Rightarrow \frac{1}{x}dx - \frac{1}{y}dy = dx$

integrating on both sides, we get

$\log x - \log y = x + \log c$

$\Rightarrow \log\left(\frac{x}{yc}\right) = x$

$\Rightarrow \frac{x}{yc} = e^x \Rightarrow \boxed{\frac{x}{y} = e^x c}$

16 $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

$\Rightarrow \frac{\sec^2 y}{\tan y}dy = -\frac{\sec^2 x}{\tan x}dx$

$\Rightarrow \log(\tan y) = -\log(\tan x) + \log c$

$\Rightarrow \log(\tan x \cdot \tan y) = \log c$

$\Rightarrow \boxed{\tan x \cdot \tan y = c}$

17 $\log\left(\frac{dy}{dx}\right) = ax + by$

$\Rightarrow \frac{dy}{dx} = e^{ax} \cdot e^{by}$

$$\Rightarrow \frac{dy}{e^{by}} = e^{ax} dx$$

Integrating on both sides, we get,

$$\boxed{\frac{e^{-by}}{-b} = \frac{e^{ax}}{a} + e^{C}}$$

8 $3e^{x} \tan y \, dx + (1-e^{x}) \sec^2 y \, dy = 0$.

Sol

$$\frac{3e^{x}}{1-e^{x}} dx + \frac{\sec^2 y}{\tan y} dy = 0.$$

Integrating on both sides, we get,

$$-3\log(1-e^{x}) + \log(\tan y) = \log C.$$

$$\Rightarrow \boxed{\frac{\tan y}{(1-e^{x})^3} = C.}$$

9 $\left(y - x\frac{dy}{dx}\right)x = y$

$$\Rightarrow xy - x^2\frac{dy}{dx} = y$$

$$\Rightarrow (x-1)y = x^2\frac{dy}{dx}$$

$$\Rightarrow \frac{x-1}{x^2} dx = \frac{1}{y} dy$$

$$\Rightarrow \frac{1}{y} dy = \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$$

$$\Rightarrow \log y = \log x + \frac{1}{x} + \log c.$$

$$\Rightarrow \boxed{y = xc\, e^{1/x}}$$

HW

10 $(x^2 - yx^2) dy + (y^2 + xy^2) dx = 0$

11 $\frac{dy}{dx} = xy + x + y + 1$.

Sol

$$\Rightarrow \frac{dy}{dx} = x(y+1) + (y+1)$$
$$= (y+1)(x+1)$$

$$\Rightarrow \frac{dy}{(y+1)} = (x+1)dx$$

$$\boxed{\log(y+1) = \frac{x^2}{2} + x + c}$$

12. $x(1+y^2)dx + y(1+x^2)dy = 0$

$$\Rightarrow \frac{x}{1+x^2}dx + \frac{y}{1+y^2}dy = 0$$

$$\Rightarrow \frac{\log(1+x^2)}{2} + \frac{\log(1+y^2)}{2} = \log c$$

$$\Rightarrow \log\left[(x^2+1)(y^2+1)\right] = \log c$$

$$\Rightarrow \boxed{(x^2+1)(y^2+1) = c}$$

13. $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$

$$\Rightarrow \frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0$$

Integrating on both sides, we get

$$\boxed{\sin^{-1}y + \sin^{-1}x = \sin^{-1}c}$$

which is the reqd. g.s.

14. Find the equation of the curve passing through the point $(1,1)$ whose diff. equation is $(y - yx)dx + (x + xy)dy = 0$ —— ①

Sol $y(1-x)dx + x(1+y)dy = 0$

$$\Rightarrow \frac{1+y}{y}dy + \frac{1-x}{x}dx = 0$$

Integrating, we get

$$\log y + \log x - x + y = c$$

$$\Rightarrow \boxed{\log(xy) + (y-x) = c} \quad \text{(2)}$$

which is reqd eqn of the curve.

Since the pt (1,1), passing through (1), we get

$$\log(1 \cdot 1) + (1-1) = c$$

$$\Rightarrow \boxed{c = 0}$$

$\therefore$ from (2), we have

$$\boxed{\log(xy) + (y-x) = 0}$$

HW 15 Find the equation of the curve that passes through the point (1,2) and satisfies the equation $\frac{dy}{dx} = \frac{-2xy}{x^2+1}$.

## * SECOND FORM :

**Equations reducible to the form in which variables can be seperated :**

Equations of the form $\frac{dy}{dx} = f(ax+by+c)$ (1) can be reduced to the form in which the variables are seperable.

put $ax+by+c = z$

diff. w.r.t $x$, we get,

$$a + b\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right)$$

$$\therefore (1) \equiv \frac{1}{b}\left(\frac{dz}{dx} - a\right) = f(z)$$

$$\Rightarrow \frac{dz}{(a+bf(z))} = dx$$

The variables have been seperated.

Integrating on both sides, we get solution.

Problems

$\rightarrow$ solve the following diff. equations:

1. $\frac{dy}{dx} = (3x+y+4)^2$ ——— ①

put $3x+y+4 = z$
diff. w.r.t. $x$, we get

$3+y' = \frac{dz}{dx}$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{dz}{dx} - 3}$$

$\therefore$ from ①, we have,

$$\frac{dz}{dx} - 3 = z^2$$

$$\Rightarrow \frac{dz}{dx} = z^2+3$$

$$\Rightarrow \frac{dz}{z^2+(\sqrt{3})^2} = dx$$

Integrating on both sides, we get,

$$\Rightarrow \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{z}{\sqrt{3}}\right) = x+C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}(x/a)$$

$$\Rightarrow \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{3x+y+4}{\sqrt{3}}\right) = x+C$$

$$\Rightarrow \tan^{-1}\left(\frac{3x+y+4}{\sqrt{3}}\right) = \sqrt{3}(x+C)$$

$$\Rightarrow \ldots = \ldots \tan[(\sqrt{3})(x+C)].$$

(15)

[2] $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$ —— (1)

Sol put $x+y = z$

$$1 + \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{dz}{dx} - 1}$$

∴ from (1), we have,

$$\frac{dz}{dx} - 1 = \cos z + \sin z$$

$$\Rightarrow \frac{dz}{dx} = (1 + \cos z) + \sin z$$

$$= 2\cos^2(z/2) + 2\sin z/2 \cos z/2$$

$$= 2\cos^2(z/2)\left[1 + \tan(z/2)\right]$$

$$\Rightarrow \frac{dz}{2\cos^2(z/2)(1+\tan(z/2))} = dx$$

$$\Rightarrow \int \frac{\sec^2(z/2)}{2[1+\tan(z/2)]} dz = \int dx$$

$$\Rightarrow \log\left[1 + \tan(z/2)\right] = x + C$$

$$\Rightarrow \boxed{\log\left[1 + \tan\left(\frac{x+y}{2}\right)\right] = x + C}$$

[3] $\frac{dy}{dx} + 1 = e^{x+y}$

$$\Rightarrow \frac{dy}{dx} = e^{x+y} - 1 \quad \text{—— (1)}$$

put $x+y = z$

$$\boxed{\frac{dy}{dx} = \frac{dz}{dx} - 1}$$

$\therefore$ (1) $\equiv$ $\frac{dz}{dx} - 1 = e^{z} - 1$

$\Rightarrow \frac{dz}{dx} = e^{z}$

$\Rightarrow -e^{-z} = x + C$

$\Rightarrow -e^{-(x+y)} = x + C$,

$\Rightarrow \boxed{e^{-(x+y)} + x + C = 0}$

[4] $(e^{y}+1)\cos x\, dx + e^{y}\sin x\, dy = 0$

$\Rightarrow \frac{\cos x}{\sin x} dx + \frac{e^{y}}{e^{y}+1} dy = 0$

$\Rightarrow \log(\sin x) + \log(e^{y}+1) = \log c$

$\Rightarrow \boxed{(\sin x)(e^{y}+1) = C}$

[5] $\frac{dy}{dx} = \sec(x+y)$ ——— (1)

put $x+y = z$

$\boxed{\frac{dy}{dx} = \frac{dz}{dx} - 1}$

$\therefore$ (1) $\equiv$ $\frac{dz}{dx} - 1 = \sec z$.

$\frac{dz}{1+\sec z} = dx$

$\Rightarrow \left(\frac{\cos z}{1+\cos z}\right) dz = dx$

$\Rightarrow \left(1 - \frac{1}{1+\cos z}\right) dz = dx$

$\Rightarrow \left(1 - \frac{1}{2\cos^2 z/2}\right) dz = dx$

$\Rightarrow \left(1 - \frac{1}{2}\sec^2 z/2\right) dz = dx$

$$\Rightarrow z - \tan\left(\frac{z}{2}\right) = x + c$$

$$\Rightarrow x+y - \tan\left(\frac{x+y}{2}\right) = x + c$$

$$\Rightarrow \boxed{y - \tan\left(\frac{x+y}{2}\right) = c}$$

## * Third Form :—

Differential eqns of the form

$$\frac{dy}{dx} = \frac{(ax+by)+c}{m(ax+by)+c_1} \quad (\text{or}) \quad \frac{dy}{dx} = \frac{m(ax+by)+c}{ax+by+c_1} \quad \text{——— (1)}.$$

put $ax+by = z$

$$\Rightarrow a + by' = z'$$

$$\Rightarrow y' = \frac{1}{b}(z' - a)$$

$$\therefore (1) \equiv \frac{1}{b}\left(\frac{dz}{dx} - a\right) = \frac{z+c}{mz+c_1}$$

$$\Rightarrow \frac{dz}{dx} = \frac{b(z+c)}{mz+c_1} + a$$

Now seperate the variables, which can be easily solved.

**problems**

→ solve the following diff. eqns:

[1] $\frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$

Sol $\frac{dy}{dx} = \frac{x-y+3}{2(x-y)+5}$ ——— (1)

put $x - y = z$

$$\frac{dy}{dx} = 1 - \frac{dz}{dx}$$

$\therefore$ ① $\equiv$ $1 - \frac{dz}{dx} = \frac{z+3}{2z+5}$

$\Rightarrow \frac{dz}{dx} = 1 - \frac{z+3}{2z+5}$

$\Rightarrow \frac{dz}{dx} = \frac{z+2}{2z+5}$

$\Rightarrow \frac{2z+5}{z+2}\, dz = dx$

$\Rightarrow \left(2 + \frac{1}{z+2}\right) dz = dx$

$$z+2 \overline{)\, 2z+5\, (} \quad 2$$
$$2z+4$$
$$1 \, (R)$$

$Q + \frac{R}{D}$

Integrating on both sides, we get

$\Rightarrow 2z + \log(z+2) = x + c$

$\Rightarrow 2(x-y) + \log(x-y+2) = x+c$

$\Rightarrow \boxed{x - 2y + \log(x-y+2) = c}$

[2] $\frac{dy}{dx} = \frac{2y-x-3}{2x-4y+5}$

$= \frac{2y-x-3}{2(x-2y)+5}$ —— ①

put $2y - x = z$

$\frac{dy}{dx} = \frac{1}{2}\left[\frac{dz}{dx} + 1\right]$

$\therefore$ ① $\equiv$ $\frac{1}{2}\left[\frac{dz}{dx} + 1\right] = \frac{z-3}{-2z+5}$

$$4z-11 \overline{)\, -2z+5\, (} \quad \tfrac{1}{2}$$
$$+2z - \tfrac{11}{2}$$
$$-\tfrac{1}{2}$$

$\Rightarrow \frac{dz}{dx} = \frac{4z-11}{-2z+5}$

$\Rightarrow \left[\frac{-2z+5}{4z-11}\right] dz = dx$

$\Rightarrow \int \left[-\frac{1}{2} - \frac{1}{2}\frac{1}{(4z-11)}\right] dz = \int dx + c$

$$\Rightarrow -\tfrac{1}{2}z - \tfrac{1}{2}\cdot\frac{\log(4z-11)}{4} = x + C$$

$$\Rightarrow \boxed{-\tfrac{1}{2}[2y-x] - \tfrac{1}{8}\log[4(2y-x)-11] = x + C}$$

which is the reqd eqn

## * Homogeneous Differential eqns:

### * Homogeneous function :-

A function $f(x,y)$ is said to be a homogeneous function of degree '$n$' in $x$ & $y$, if $f(kx,ky) = k^n f(x,y)$ $\forall$ $n$ & $k$ is const.

Ex: (1) $f(x,y) = \dfrac{x^2+y^2}{x^3+y^3} \Rightarrow f(kx,ky) = \dfrac{k^2x^2+k^2y^2}{k^3x^3+k^3y^3}$

$$= k^{-1}\frac{(x^2+y^2)}{(x^3+y^3)}$$

$$= k^{-1} f(x,y).$$

$\therefore$ $f(x,y)$ is a homogeneous function in $x$ & $y$, of degree $-1$

(2) $f(x,y) = \dfrac{\sqrt[3]{x}+\sqrt[3]{y}}{x+y} \Rightarrow f(kx,ky) = \dfrac{\sqrt[3]{kx}+\sqrt[3]{ky}}{kx+ky}$

$$= k^{\frac{1}{3}-1}\left(\frac{\sqrt[3]{x}+\sqrt[3]{y}}{x+y}\right)$$

$$= k^{-2/3} f(x,y).$$

$\therefore$ $f(x,y)$ is homo. function of degree $-2/3$.

(3) $f(x,y) = \cos x + \tan y$
$\Rightarrow f(kx,ky) = \cos kx + \tan ky$
$\neq k^n f(x,y)$.

Hence it is not homogeneous function.

(4) $f(x,y) = \dfrac{(\sqrt{x}+\sqrt{y})}{\sqrt{x}-\sqrt{y}}$

$\Rightarrow f(kx,ky) = k^0 f(x,y)$

$\therefore f(x,y)$ is a homogeneous fn. of degree zero in x & y

Note:- If $f(x,y)$ is a homogeneous fn of degree zero then $f(x,y)$ is a function of $y/x$ (or) $\frac{x}{y}$.

* **Homogeneous Diff. equation:-**

A diff. eqn is said to be homogeneous, if it can be put in the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$

where f, g are homogeneous functions of same degree in x & y.

**Working rule:-**

step(1): put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

step(2): put the above values in the given diff. eqn

step(3): separate the variables and integrate.

step(4): Replace v by $y/x$ to get the reqd solution.

Problems:

## Problems

*- Solve the following diff. eqns

→ $x^2 y\,dx - (x^3+y^3)\,dy = 0$.

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 y}{x^3+y^3} \quad \text{—(1)}$$

clearly (1) is homo. diff. eqn.

put $y = vx$

$$y' = v + x\frac{dv}{dx}$$

$$(1) \equiv v + x\frac{dv}{dx} = \frac{x^2 vx}{x^3+v^3x^3}$$

$$= \frac{v}{1+v^3}$$

$$\therefore x\frac{dv}{dx} = \frac{\cancel{v} - \cancel{v} - v^4}{1+v^3}$$

$$\Rightarrow \frac{1+v^3}{v^4}\,dv = -\frac{1}{x}\,dx$$

$$\Rightarrow \int\left(\frac{1}{v^4}+\frac{1}{v}\right)dv = -\int \frac{1}{x}\,dx + c$$

$$\Rightarrow -\frac{1}{3v^3} + \log v = -\log x + c.$$

$$\Rightarrow -\frac{1}{3}\left(\frac{x^3}{y^3}\right) + \log\left(\frac{y}{x}\right) + \log x = c \qquad \left(\because \frac{y}{x} = v\right)$$

$$\Rightarrow \boxed{-\frac{1}{3}\left(\frac{x^3}{y^3}\right) + \log y = c}$$

→ $(1+e^{x/y})\,dx + e^{x/y}(1 - x/y)\,dy = 0$

Sol
$$\frac{dx}{dy} = \frac{-e^{x/y}(1-x/y)}{1+e^{x/y}} \quad \text{—(1)}$$

put $x = vy$

$$\Rightarrow \boxed{x' = v + y\frac{dv}{dy}}$$

$$(1) \equiv v + y\frac{dv}{dy} = \frac{-e^{v}(1-v)}{1+e^{v}}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^{v} \cancel{+ve^{v}} - v \cancel{- ve^{v}}}{1+e^{v}}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-[v+e^{v}]}{1+e^{v}}$$

$$\Rightarrow \int\left[\frac{1+e^{v}}{v+e^{v}}\right]dv = -\int \frac{1}{y}\,dy + \log c$$

$$\Rightarrow \log(v+e^{v}) = -\log y + \log c$$

$$\Rightarrow \log\left(\frac{x}{y} + e^{x/y}\right) = -\log y + \log c \qquad \left(\because \frac{x}{y} = v\right)$$

$$\Rightarrow \log\left[\left(\frac{x}{y} + e^{x/y}\right)y\right] = \log c$$

$$\Rightarrow \boxed{x + y e^{x/y} = c}$$

→ $\dfrac{dy}{dx} = \dfrac{y}{x} + e^{y/x}$ —(1)

put $y = vx$

$$\Rightarrow y' = v + x\frac{dv}{dx}$$

$$(1) \equiv v + x\frac{dv}{dx} = v + e^{v}$$

$$\Rightarrow x\frac{dv}{dx} = e^{v}$$

$$\Rightarrow \int e^{-v}\,dv = \int \frac{1}{x}\,dx + c$$

$$= -e^{-v} = \log x + c$$

$$\Rightarrow \boxed{-e^{-y/x} = \log x + c}$$

→ $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$ —(1)

put $y = vx \Rightarrow y' = v + x\dfrac{dv}{dx}$

$$(1) \equiv v + x\frac{dv}{dx} = \frac{x+vx}{x-vx}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v}{1-v} - v$$

$$= \frac{1+\cancel{v}-\cancel{v}+v^2}{1-v}$$

$$= 1+v^2$$

$\Rightarrow \frac{1-v}{1+v^2}\,dv = \frac{1}{x}\,dx$

$\Rightarrow \int\left(\frac{1}{1+v^2} - \frac{v}{1+v^2}\right)dv = \int \frac{1}{x}\,dx + c$

$\Rightarrow \tan^{-1} v - \frac{1}{2}\log(1+v^2) = \log x + c$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \log\left[\left(\sqrt{1+v^2}\right)\cdot x\right] + c$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \log\left[\sqrt{\left(1+\frac{y^2}{x^2}\right)}\,x\right] + c.$

## Non-Homogeneous Diff. eqns:-

(or)

## Eqns Reducible to Homo form:-

Diff. eqns of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1} \quad \text{——— (1)}$$

**Case (i)** $\frac{a}{a_1} \neq \frac{b}{b_1}$ i.e $ab_1 - a_1b \neq 0$

**Working rule:**

put $x = X+h$ & $y = Y+k$ where h & k are constants

$dx = dX$ ; $dy = dY$

$\Rightarrow \frac{dy}{dx} = \frac{dY}{dX}$

$(1) \equiv \frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a_1(X+h)+b_1(Y+k)+c_1}$

$= \frac{aX+bY+(ah+bk+c)}{a_1X+b_1Y+(a_1h+b_1k+c_1)}$ ——→ (2)

choosing h & k s.t.

$ah+bk+c=0$

$a_1h+b_1k+c_1=0$

solving these equations ~~we get~~ we ~~the values~~ of h & K

i.e $\frac{h}{bc_1-b_1c} = \frac{k}{ca_1-ac_1} = \frac{1}{ab_1-a_1b}$

$\Rightarrow h = \frac{bc_1-b_1c}{ab_1-a_1b}$ & $K = \frac{ca_1-ac_1}{ab_1-a_1b}$

$(2) \equiv \frac{dY}{dX} = \frac{aX+bY}{a_1X+b_1Y}$

which is clearly homogeneous.

This eqn can be solved by putting $Y = VX$

Finally replacing Y by y-k and X by x-h.

**Case (ii):** $\frac{a}{a_1} = \frac{b}{b_1}$ i.e $ab_1 - a_1b = 0$

∴ h & K both become infinite.

Hence the method fails.

Now $\frac{a}{a_1} = \frac{b}{b_1} = \frac{1}{m}$ (say)

$\Rightarrow a_1 = am$ ; $b_1 = bm$

$\therefore (1) \equiv \frac{dy}{dx} = \frac{ax+by+c}{max+mby+c_1}$

$= \frac{(ax+by)+c}{m(ax+by)+c_1}$

put $ax+by = z$

This can be easily solved by variable seperable

## Problems

$$\rightarrow \frac{dy}{dx} = \frac{y+x-2}{y-x-4} \quad \text{—(1)}$$

$\frac{1}{1} \neq \frac{1}{-1}$ i.e $\frac{a}{a_1} \neq \frac{b}{b_1}$..

clearly (1) is not homo.

put $x = X+h$ ; $y = Y+k$

$dx = dX$ ; $dy = dY$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{dY}{dX}}$$

$$\therefore (1) \equiv \frac{dY}{dX} = \frac{Y+k+X+h-2}{Y+k-X-h-4}$$

$$= \frac{(X+Y)+(h+k-2)}{(-X+Y)+(-h+k-4)} \quad \text{—(2)}$$

choosing h, k s.t

$h+k-2=0$ —(3)

$-h+k-4=0$ —(4)

$2k = 6$

$\Rightarrow \boxed{k=3}$

$\boxed{h=-1}$

$$\therefore (2) \equiv \frac{dY}{dX} = \frac{X+Y}{-X+Y} \rightarrow (5)$$

put $Y = vX$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$\therefore (5) \equiv v + X\frac{dv}{dX} = \frac{X(1+v)}{X(-1+v)}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{1+v}{-1+v} - v$$

$$\Rightarrow X\frac{dv}{dX} = \frac{1+v+v-v^2}{-1+v}$$

$$= \frac{1+2v-v^2}{v-1}$$

$$\Rightarrow \frac{v-1}{1+2v-v^2}dv = \frac{1}{X}dX$$

$$\Rightarrow \int\left[\frac{-\frac{1}{2}(-2v+2)}{-v^2+2v+1}\right]dv = \int\frac{1}{X}dX + c_1$$

$$\Rightarrow -\frac{1}{2}\log[-v^2+2v+1] = \log X + \log c$$

$$\Rightarrow \log\left(\frac{1}{[-v^2+2v+1]^{1/2}}\right) = \log(Xc)$$

$$\Rightarrow \frac{1}{\left[-\frac{Y^2}{X^2}+2\frac{Y}{X}+1\right]^{1/2}} = Xc$$

$$\Rightarrow \frac{X}{[-Y^2+2YX+X^2]^{1/2}} = Xc$$

$$\Rightarrow \frac{1}{[-(y-k)^2+2(y-k)(x-h)+(x-h)^2]^{1/2}} = c$$

$$\Rightarrow \frac{1}{[-(y-3)^2+2(y-3)(x+1)+(x+1)^2]^{1/2}} = c$$

H.W $\rightarrow \frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$

93 [6] $\left\{y\left(1+\frac{1}{x}\right)+\cos y\right\}dx+\left(x+\log x-x\sin y\right)dy=0$ — (1)

Sol$^n$ $M=y\left(1+\frac{1}{x}\right)+\cos y$ ; $N=x+\log x-x\sin y$

$\frac{\partial M}{\partial y}=1+\frac{1}{x}-\sin y$ ; $\frac{\partial N}{\partial x}=1+\frac{1}{x}-x\sin y$

$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

$\therefore$ G.S. of (1) is

$$\int\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+\int 0\,dy=\int 0$$

$$\Rightarrow \boxed{y(x+\log x)+x\cos y=c}$$

[7]. $(2ax+by)y\,dx+(ax+2by)x\,dy=0$

Sol$^n$ $M=2axy+by^2$ ; $N=ax^2+2byx$

$\frac{\partial M}{\partial y}=2ax+2by$ ; $\frac{\partial N}{\partial x}=2ax+2by$.

$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

G.S. is $\int(2axy+by^2)dx+\int 0\,dy=\int 0$

$$\Rightarrow \boxed{ax^2y+bxy^2=c}$$

[8]. $(x^2-ay)dx-(ax-y^2)dy=0$

Sol$^n$ $M=x^2-ay$ ; $N=-(ax-y^2)$

$\frac{\partial M}{\partial y}=-a$ ; $\frac{\partial N}{\partial x}=-a$

$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

G.S. is $\int(x^2-ay)dx+\int y^2dy=\int 0$

$$\Rightarrow \frac{x^3}{3}-axy+\frac{y^3}{3}=c_1$$

$$\Rightarrow \boxed{x^3-3axy+y^3=c} \quad \text{where } c=3c_1$$

10. $\frac{dy}{dx} = \frac{2x-y}{x+2y-5}$

Sol:

$\Rightarrow (2x-y)\,dx \quad (x+2y-5)\,dy = 0$ —— (1)

$M = 2x-y \;;\; N = -(x+2y-5)$

$\frac{\partial M}{\partial y} = -1 \;;\; \frac{\partial N}{\partial x} = -1$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

G.S is $\int (2x-y)\,dx - \int (2y-5)\,dy = \int 0$

$\Rightarrow \boxed{x^2 - xy - y^2 + 5y = c}$

HW 11. $(x^2+y^2+a^2)y\,dy + (x^2+y^2-a^2)x\,dx = 0$

HW 12. $x(x^2+3y^2)\,dx + y(y^2+3x^2)\,dy = 0$

HW 13. $(a^2-2xy-y^2)\,dx - (x+y)^2\,dy = 0$

14. $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

Sol: $M = 1+e^{x/y} \;;\; N = e^{x/y}\left(1-\frac{x}{y}\right)$

$\frac{\partial M}{\partial y} = e^{x/y}\left(-\frac{x}{y^2}\right) \;;\; \frac{\partial N}{\partial x} = e^{x/y}\left(-\frac{1}{y}\right) + \left(1-\frac{x}{y}\right)e^{x/y}\cdot\frac{1}{y}$

$= -e^{x/y}\cdot\frac{x}{y^2}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\therefore$ G.S. is

$\int \left(1+e^{x/y}\right)dx + \int 0\,dy = \int 0$

$\Rightarrow x + \frac{e^{x/y}}{1/y} = c$

$\Rightarrow \boxed{x + y\,e^{x/y} = c}$

# Exact Differential equations:

**Defn.** The diff. equation $M(x,y)dx + N(x,y)dy = 0$ where M & N are functions of x & y, is called an exact diff. equation if $Mdx + Ndy = 0$ is an exact derivative of x & y.

i.e, $Mdx + Ndy = du$, where u is a function of x & y.

$$Mdx + Ndy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

Ex (1) $x\,dy + y\,dx = 0$ is an exact.

Because $x\,dy + y\,dx = d(xy)$

$$\Rightarrow x\,dy + y\,dx = \frac{\partial}{\partial x}(xy)\,dx + \frac{\partial}{\partial y}(xy)\,dy$$

$xy = c$
$\frac{d}{dx}(xy) = 0$
$x\frac{dy}{dx} + y = 0$
$x\,dy + y\,dx = 0$
$\therefore d(xy) = x\,dy + y\,dx$

(2) $\frac{1}{x}dy - \frac{y}{x^2}dx = 0$ is an exact.

Because $\frac{1}{x}dy - \frac{y}{x^2}dx = \frac{x\,dy - y\,dx}{x^2} = d(y/x)$

**Note:** The diff. eqn $Mdx + Ndy = 0$ is an exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

**working rule:**

(1) The diff. eqn $Mdx + Ndy = 0$ is an exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(2) The G.S. is

$$\int M\,dx + \int (\text{terms in } N \text{ not containing } x)\,dy = c$$

y-constant — ~~put x=0~~ ⇓ Drop the terms containing x

**Problems** Solve the following diff-equations.

(1) $(x+2y-2)dx + (2x-y+3)dy = 0$ —— ①

Comparing ① with $Mdx + Ndy = 0$

we have $\frac{\partial M}{\partial y} = 2$ ; $\frac{\partial N}{\partial x} = 2$.

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

G.S is $\int (x+2y-2)\,dx + \int (-y+3)\,dy = \int 0$

$$\Rightarrow \frac{x^2}{2} + 2xy - 2x - \frac{y^2}{2} + 3y = c.$$

(2) $\frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$

$\Rightarrow (ax+hy+g)dx + (hx+by+f)dy = 0$ —— ①

$\frac{\partial M}{\partial y} = h$ ; $\frac{\partial N}{\partial x} = h$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

G.S of ① is

$\int (ax+hy+g)dx + \int (by+f)dy = \int 0$

$\Rightarrow \frac{ax^2}{2} + hxy + gx + \frac{by^2}{2} + fy = c$

$\Rightarrow ax^2 + by^2 + 2gx + 2fy + 2hxy = 2c$.

HW $\boxed{3}$ $(y^2 e^{xy^2} + 4x^3)dx + (2xy e^{xy^2} - 3y^2)dy = 0$ $(e^{xy^2} + x^4 - y^3 = c)$.

$\boxed{4}$ $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

Sol$^n$: $(e^y+1)\cos x\,dx + e^y \sin x\,dy = d\left[(e^y+1)\sin x\right]$

$\therefore$ G.S. is $(e^y+1)\sin x = c$

96 $\boxed{5}$. $y\sin 2x\,dx - (y^2+\cos^2 x+1)dy = 0$ —— ①

Sol$^n$: $M = y\sin 2x$ ; $N = -y^2 + \cos^2 x + 1$

$\Rightarrow \frac{\partial M}{\partial y} = \sin 2x$ ; $\frac{\partial N}{\partial x} = 2\cos x \sin x = \sin 2x$.

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

G.S. of ① is

$\int y\sin 2x\,dx - \int (y^2+1)dy = \int 0$

$\Rightarrow -y\frac{\cos 2x}{2} - \frac{y^3}{3} - y = c_1$

$\Rightarrow \frac{y\cos 2x}{2} + \frac{y^3}{3} + y = C$ where $C = -c_1$

## Integrating factor:

Sometimes $M\,dx + N\,dy = 0$ is not exact but it can be made exact by multiplying throughout by a suitable non-zero function $\mu(x, y)$. This multiplier is called the integrating factor.

**Note:** If the given diff. eqn can be transformed into the following formulas then the equations are exact.

(1) $d(xy) = x\,dy + y\,dx$

(2) $d(y/x) = \dfrac{x\,dy - y\,dx}{x^2}$

(3) $d(x/y) = \dfrac{y\,dx - x\,dy}{y^2}$

(4) $d(y^2/x) = \dfrac{2xy\,dy - y^2\,dx}{x^2}$

(5) $d(x^2/y) = \dfrac{2xy\,dx - x^2\,dy}{y^2}$

(6) $d(y^2/x^2) = \dfrac{2x^2y\,dy - 2xy^2\,dx}{x^4} = \dfrac{2xy(x\,dy - y\,dx)}{x^4}$

(7) $d\left(\dfrac{x^2}{y^2}\right) = \dfrac{2y^2x\,dx - 2x^2y\,dy}{y^4} = \dfrac{2xy(y\,dx - x\,dy)}{y^4}$

(8) $d(e^y/x) = \dfrac{xe^y\,dy - e^y\,dx}{x^2}$

(9) $d(e^x/y) = \dfrac{ye^x\,dx - e^x\,dy}{y^2}$

(10) $d[\log|xy|] = \dfrac{x\,dy + y\,dx}{xy}$

(11) $d\left[\log\left|\dfrac{y}{x}\right|\right] = \dfrac{x\,dy - y\,dx}{xy}$

(12) $d\left[\log\left|\dfrac{x}{y}\right|\right] = \dfrac{y\,dx - x\,dy}{xy}$

(13) $d\left(\tan^{-1}\dfrac{y}{x}\right) = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$

(14) $d\left(\tan^{-1}\dfrac{x}{y}\right) = \dfrac{y\,dx - x\,dy}{x^2 + y^2}$

(15) $d\left[\log\sqrt{x^2+y^2}\right]$ (or) $d\left[\frac{1}{2}\log(x^2+y^2)\right] = \dfrac{x\,dx + y\,dy}{x^2+y^2}$

(16) $d\left(-\dfrac{1}{xy}\right) = \dfrac{x\,dy + y\,dx}{x^2y^2}$

Ex:① $2y\,dx + x\,dy = 0$ —— ①

$\therefore$ $M = 2y$ ; $N = x$

$\frac{\partial M}{\partial y} = 2$ ; $\frac{\partial N}{\partial x} = 1$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Multiplying ① with $x$, we get

$2xy\,dx + x^2 dy = 0$ —— ②

$\frac{\partial M}{\partial y} = 2x$ ; $\frac{\partial N}{\partial x} = 2x$.

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\therefore$ eqn ② is an exact.

$\therefore$ $x$ is an integrating factor of

$2y\,dx + x\,dy = 0$

and $2xy\,dx + x^2 dy = d(x^2 y)$.

Ex:-② $y\,dx - x\,dy = 0$ —— ①

$\frac{\partial M}{\partial y} = 1$ ; $\frac{\partial N}{\partial x} = -1$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

$\therefore$ ① is not exact

Multiplying ① by $\frac{1}{x^2}$ we get,

$\frac{y}{x^2} dx - \frac{1}{x} dy = 0$ —— ②

$\frac{\partial M}{\partial y} = \frac{1}{x^2}$ ; $\frac{\partial N}{\partial x} = \frac{1}{x^2}$.

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\therefore$ ② is an exact.

$\therefore \frac{1}{x^2}$ is I.F. of ①.

and $-\frac{y}{x^2} dx + \frac{1}{x} dy = 0$

$\Rightarrow \frac{-y\,dx + x\,dy}{x^2} = d(y/x)$

[4]. $y\,dx - x\,dy + (1+x^2)\,dx + x^2 \sin y\,dy = 0$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2} - \left(\frac{1}{x^2}+1\right)dx - \sin y\,dy = 0$$

$$\Rightarrow d(y/x) - \left(\frac{1}{x^2}+1\right)dx - \sin y\,dy = 0$$

Integrating, we get

$$\boxed{y/x - \left(x - \frac{1}{x}\right) + \cos y = c}$$

2nd [5]. $y \sin 2x\,dx = (1+y^2+\cos^2 x)\,dy$

$$\Rightarrow \cos^2 x\,dy - 2y \sin x \cos x\,dx + (1+y^2)\,dy = 0$$

$$\Rightarrow d(y \cos^2 x) + (1+y^2)\,dy = 0$$

Integrating, we get

$$\boxed{y\cos^2 x + y + \frac{y^3}{3} = c}$$

[6]. $y(2x^2y + e^x)\,dx - (e^x + y^3)\,dy = 0$

$$\Rightarrow e^x y\,dx - e^x\,dy + 2x^2y^2\,dx - y^3\,dy = 0$$

$$\Rightarrow \frac{e^x y\,dx - e^x\,dy}{y^2} + 2x^2\,dx - y\,dy = 0$$

$$\Rightarrow d(e^x/y) + 2x^2\,dx - y\,dy = 0$$

Integrating, we get

$$\boxed{e^x/y + \frac{2}{3}x^3 - \frac{y^2}{2} = c}$$

2nd [7]. $x\,dy = [y + x\cos^2(y/x)]\,dx$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2} = \frac{1}{x}\cos^2(y/x)\,dx$$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2 \cos^2 y/x} = \frac{1}{x}\,dx$$

$$\Rightarrow \sec^2(y/x)\left[\frac{x\,dy - y\,dx}{x^2}\right] = \frac{1}{x}\,dx$$

$$\Rightarrow d[\tan(y/x)] = \frac{1}{x}\,dx.$$

Integrating, we get $\boxed{\tan(y/x) = \log x + c}$

Also $\dfrac{y\,dx - x\,dy}{y^2} = d(x/y)$ ;

$\dfrac{y\,dx - x\,dy}{xy} = d\left[\log(x/y)\right]$ ;

and $\dfrac{y\,dx - x\,dy}{x^2+y^2} = d\left(\tan^{-1}(x/y)\right)$

$\therefore$ $\dfrac{1}{y^2}, \dfrac{1}{xy}, \dfrac{1}{x^2+y^2}$ are Integrating factors of $y\,dx - x\,dy = 0$

From the above example we observe that a diff. eqn has more than one I.F.

Problems

[1]. $x\,dy - y\,dx + 2x^3\,dx = 0$

Soln $\Rightarrow \dfrac{x\,dy - y\,dx}{x^2} + 2x\,dx = 0$

$\Rightarrow \quad d(y/x) + 2x\,dx = 0$

Integrating, we get

$$\boxed{y/x + x^2 = c}$$

[2] $\dfrac{x\,dy - y\,dx}{x^2+y^2} = x\,dx$

Soln: $\Rightarrow d\left(\tan^{-1}(y/x)\right) = x\,dx$

Integrating we get

$$\boxed{\tan^{-1}(y/x) = \frac{x^2}{2} + c}$$

[3]. $x\,dy - y\,dx = xy^2\,dx$

Soln $\Rightarrow \dfrac{y\,dx - x\,dy}{y^2} + x\,dx = 0 \Rightarrow d(x/y) + x\,dx = 0$

Integrating, we get

$$\boxed{\frac{x}{y} + \frac{x^2}{2} = c}$$

8. $\left(y+\cos y+\frac{1}{2\sqrt{x}}\right)dx+(x-x\sin y-1)\,dy=0$

$\Rightarrow (y\,dx+x\,dy)+(\cos y\,dx-x\sin y\,dy)+\frac{1}{2\sqrt{x}}dx-dy=0$

$\Rightarrow d(xy)+d(x\cos y)+\frac{1}{2\sqrt{x}}-dy=0$

Integrating

$\boxed{xy+x\cos y+\sqrt{x}-y=C}$

9. $(xy^2+2x^2y^3)dx+(x^2y-x^3y^2)dy=0$

$\Rightarrow xy^2(1+2xy)\,dx+x^2y(1-xy)\,dy=0$

$\Rightarrow y(1+2xy)\,dx+x(1-xy)\,dy=0$

$\Rightarrow (y\,dx+x\,dy)+2xy^2dx-x^2y\,dy=0$

$\Rightarrow \frac{y\,dx+x\,dy}{x^2y^2}+\frac{2}{x}dx-\frac{1}{y}dy=0$

$\Rightarrow \frac{d(xy)}{x^2y^2}+\frac{2}{x}dx-\frac{1}{y}dy=0$ —— (2)

Clearly (2) is an exact.

Integrating (2), we get

$-\frac{1}{xy}+2\log x-\log y=\log c.$

$\Rightarrow -\frac{1}{xy}+\log\left(\frac{x^2}{yc}\right)=0$

$\Rightarrow \log\left(\frac{x^2}{yc}\right)=\frac{1}{xy}$

$\Rightarrow \boxed{x^2/yc=e^{\frac{1}{xy}}}$

10. $(x^2+y^2-a^2)y\,dy+x(x^2+y^2-b^2)dx=0$ —— (1)

$\Rightarrow (x^2+y^2)[2y\,dy+2x\,dx]-2a^2y\,dy-2xb^2dx=0$

$\Rightarrow (x^2+y^2)\,d(x^2+y^2)-2a^2y\,dy-2xb^2dx=0$

Integrating.

$\int z\,dz-2a^2\int y\,dy-2b^2\int x\,dx=\int 0$ ; where $x^2+y^2=z$

$$\Rightarrow \frac{z^2}{2} - a^2y^2 - b^2x^2 = c_1$$

$$\Rightarrow \boxed{(x^2+y^2)^2 - 2a^2y^2 - 2b^2x^2 = c} \quad \text{where } c = 2c_1$$

[11]. $x\,dy - (y-x)\,dx = 0$ —— ①

$$\Rightarrow x\,dy - y\,dx + x\,dx = 0$$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2} + \frac{1}{x}dx = 0$$

$$\Rightarrow d(y/x) + \frac{1}{x}dx = 0 \quad \text{—— ②}$$

$$\Rightarrow \int d(y/x) + \int \frac{1}{x}dx = \int 0$$

$$\Rightarrow \boxed{y/x + \log x = c}$$

H.W.

[12]. $x\,dy - y\,dx = xy^2\,dx.$

[13]. $x\,dx + y\,dy + (x^2+y^2)\,dy = 0$

$$\Rightarrow \frac{x\,dx + y\,dy}{x^2+y^2} + dy = 0$$

$$\Rightarrow d\left(\log\sqrt{x^2+y^2}\right) + dy = 0$$

Integrating, we get,

$$\boxed{\log\sqrt{x^2+y^2} + y = c.}$$

[14]. $x\,dy - y\,dx = (x^2+y^2)\,dx$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2+y^2} = dx$$

$$\Rightarrow d\left(\tan^{-1}(y/x)\right) = dx$$

Integrating, we get

$$\boxed{\tan^{-1}(y/x) = x + c}$$

[15] $y\,dx + x\,dy + \log x\,dx = 0$

$$\Rightarrow \frac{-y\,dx + x\,dy}{x^2} = \frac{\log x}{x^2} \qquad \log x = t \Rightarrow \frac{1}{x}dx = dt \Rightarrow x = e^t$$

$$\int d(y/x) = \int t e^{-t}\,dt + c$$

$\Rightarrow y/x = -e^{-t}(t+1) + c$

$\Rightarrow y/x = -\frac{1}{x}(\log x + 1) + c.$

16. $(x^2+y^2-2y)\,dy = 2x\,dx.$

$\Rightarrow (x^2+y^2)\,dy = 2x\,dx + 2y\,dy$

$\Rightarrow \frac{2x\,dx + 2y\,dy}{x^2+y^2} = dy$

$\Rightarrow d(\log(x^2+y^2)) = dy$

Integrating, we get

$$\boxed{\log(x^2+y^2) = y + C}$$

H.W.

17. $y\,dx - x\,dy = 3x^2 e^{x^3} y^2\,dx.$

18. $(y - xy^2)\,dx - (x + x^2y)\,dy = 0$

$\Rightarrow y\,dx - x\,dy - xy(y\,dx + x\,dy) = 0.$

$\Rightarrow \frac{1}{x}dx - \frac{1}{y}dy - d(xy) = 0$

Integrating, we get

$$\boxed{\log x - \log y - xy = C}$$

19. $y(2xy + e^x)\,dx = e^x\,dy$

$\Rightarrow ye^x\,dx - e^x\,dy + 2xy^2\,dx = 0$

$\Rightarrow d(e^x/y) + 2x\,dx = 0$

Integrating, we get

$$\boxed{e^x/y + x^2 = C}$$

20. Solve $x^2\left(\frac{dy}{dx}\right) + xy = \sqrt{1-x^2y^2}$

$\Rightarrow x\left[\frac{x\,dy + y\,dx}{dx}\right] = \sqrt{1-(xy)^2}$

$\Rightarrow \frac{x\,dy + y\,dx}{\sqrt{1-(xy)^2}} = \frac{dx}{x}$

$\Rightarrow \frac{d(xy)}{\sqrt{1-(xy)^2}} = \frac{dx}{x}.$

Integrating, we get

$$\boxed{\sin^{-1}(xy) - \log x = C}$$

# Methods for finding integrating factors:

**Method 1:** If $Mdx+Ndy=0$ is homogeneous and $Mx+Ny\neq 0$, then $\frac{1}{Mx+Ny}$ is an I.f

**Problems**

find I.f and solve the following equations.

[1] $x^2y\,dx-(x^3+y^3)dy=0$ —— ①

Sol$^n$ Comparing ① with $Mdx+Ndy=0$

$M=x^2y$ ; $N=-(x^3+y^3)$.

$\frac{\partial M}{\partial y}=x^2$ ; $\frac{\partial N}{\partial x}=-3x^2$.

$\therefore \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$.

① is not exact

$Mx+Ny=x^3y-x^3y-y^4=-y^4\neq 0$.

$\Rightarrow \frac{1}{Mx+Ny}=-\frac{1}{y^4}$

Multiplying ① by $-\frac{1}{y^4}$, we get:

$-\frac{x^2}{y^3}dx+\left(\frac{x^3}{y^4}+\frac{1}{y}\right)dy=0$ —— ②

Comparing ② with $Pdx+Qdy=0$

$P=-x^2/y^3$ ; $Q=\frac{x^3}{y^4}+\frac{1}{y}$

$\frac{\partial P}{\partial y}=\frac{3x^2}{y^4}$ ; $\frac{\partial Q}{\partial x}=\frac{3x^2}{y^4}$.

$\therefore \frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$

$\therefore$ ② is an exact

it's solution is given by

$\int\left(-\frac{x^2}{y^3}\right)dx+\int\frac{1}{y}dy=\int 0$

$\Rightarrow \frac{-x^3}{-3y^3}+\log y=\log c$.

$\Rightarrow \log(y/c)=\frac{x^3}{3y^3}$

$\Rightarrow \boxed{y=c\,e^{x^3/3y^3}}$

[2]. $y^2 dx + (x^2 - xy - y^2) dy = 0 \to \frac{1}{y(x^2-y^2)} \to \frac{1}{2}\tan^{-1}(x/y) + \log y = c.$

[3]. $(x^2+y^2)\,dx - 2xy\,dy = 0 \to \frac{1}{x(x^2-y^2)} \to$

[4]. $xy\,dx - (x^2+2y^2)\,dy = 0$

[5]. $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

[6]. $(3x^2 - y^3)\,dx - (2x^2y - xy^2)\,dy = 0$

[7]. $(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$

Method 2: If $M dx + N dy = 0$ is such that $M = y f_1(x,y)$ and $N = x f_2(x,y)$ i.e., $y f_1(xy)\,dx + x f_2(xy)\,dy = 0$ and $Mx - Ny \neq 0$ then $\frac{1}{Mx - Ny}$ is an Integrating factor.

problems:

find I.F & solve.

[1]. $y(1+2xy)\,dx + x(1-2xy)\,dy = 0$ —— (1)

Sol: Comparing (1) with $M dx + N dy = 0$.

$M = y(1+2xy)$ ; $N = x(1-2xy)$

clearly (1) is of the form $y f_1(x,y)dx + x f_2(x,y)dy = 0$

$\therefore Mx - Ny = xy + 2x^2y^2 - xy + 2x^2y^2$

$= 4x^2y^2 \neq 0$

$\therefore I.F = \frac{1}{Mx - Ny} = \frac{1}{4x^2y^2}$

Multiplying (1) by $\frac{1}{4x^2y^2}$

$$\left(\frac{1}{4x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{4xy^2} - \frac{1}{2y}\right)dy = 0 \quad \text{—— (2)}$$

Clearly (2) is an exact.

Integrating, we get

$$-\frac{1}{4xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c_1$$

$$\Rightarrow \boxed{-\frac{1}{xy} + 2\log(x/y) = c} \quad \text{where } c = 4c_1$$

12. $(xy\sin xy+\cos xy)y\,dx+(xy\sin xy-\cos xy)x\,dy=0$

Sol: Comparing ① with $M\,dx+N\,dy=0$ — ①

$M=(xy\sin xy+\cos xy)y$ , $N=(xy\sin xy-\cos xy)x$

Clearly ① is of the form $y\,f_1(x,y)\,dx+x\,f_2(x,y)\,dy=0$

$\therefore Mx-Ny=\cancel{x^2y^2\sin xy}+xy\cos xy-\cancel{x^2y^2\sin xy}+xy\cos xy$

$=2xy\cos xy\neq 0$

$\therefore$ I.F $=\dfrac{1}{Mx-Ny}=\dfrac{1}{2xy\cos xy}$

Multiplying ① by $\dfrac{1}{2xy\cos xy}$

$\left(y\tan xy+\frac{1}{x}\right)dx+\left(x\tan xy-\frac{1}{y}\right)dy=0$ — ②

Clearly ② is an exact.

Integrating, we get

$y\dfrac{\log|\sec xy|}{y}+\log x-\log y=\log c$

$\Rightarrow \log|\sec xy|+\log\frac{x}{y}=\log c$

$\Rightarrow \log\left|\frac{x}{y}\sec xy\right|=\log c$

$\Rightarrow \frac{x}{y}\sec xy=c$

$\Rightarrow \boxed{x\sec xy=cy}$

HW

3. $(xy^2+2x^2y^3)\,dx+(x^2y-x^3y^2)\,dy=0$

4. $(x^2y^2+xy+1)y\,dx+(x^2y^2-xy+1)x\,dy=0$

5. $y(1-xy)\,dx-x(1+xy)\,dy=0$

6. $y(x^2y^2+2)\,dx+x(2-2x^2y^2)\,dy=0$

7. $y(1+xy)\,dx+x(1-xy)\,dy=0$

8. $(x^4y^4+x^2y^2+xy)y\,dx+(x^4y^4-x^2y^2+xy)x\,dy=0$

**Method 3:** If $M\,dx + N\,dy = 0$ is such that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \text{ or } k \text{ (constant)}$$

then $I.F = e^{\int f(x)\,dx}$ or $e^{\int k\,dx}$.

**Problems:**

Find I.F. & solve the following diff. eqns.

1. $(x^2+y^2+2x)\,dx + 2y\,dy = 0$ — (1)

Sol$^n$ $M = x^2+y^2+2x$ ; $N = 2y$

$\frac{\partial M}{\partial y} = 2y$ ; $\frac{\partial N}{\partial x} = 0$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Now $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2y}(2y) = 1$ (constant)

$I.F. = e^{\int dx} = e^x$.

Multiplying (1) by $e^x$.

$(x^2e^x + e^xy^2 + 2xe^x)\,dx + 2ye^x\,dy = 0$ — (2)

Clearly (2) is an exact.

$\int e^x(x^2+y^2+2x)\,dx = \int 0$

$\Rightarrow (x^2+y^2+2x)e^x - \int (2x+2)e^x\,dx = c$

$\Rightarrow (x^2+y^2+2x)e^x - 2[e^x(x-1)+e^x] = c$

$\Rightarrow (x^2+y^2+2x)e^x - 2xe^x = c$

$\Rightarrow \boxed{e^x(x^2+y^2) = c}$

(or) $\frac{2x\,dx + 2y\,dy}{x^2+y^2} + dx = 0$

$d[\log(x^2+y^2)] + dx$

Integrating.

$\log(x^2+y^2) + x = \log c \Rightarrow \frac{x^2+y^2}{c} = e^{-x}$

$\Rightarrow e^x(x^2+y^2) = c$

2. $\left(y+\frac{1}{3}y^3+\frac{1}{2}x^2\right)dx+\frac{1}{4}(x+xy^2)dy=0$ ——(1)

$M = y+\frac{1}{3}y^3+\frac{1}{2}x^2$ ; $N = \frac{1}{4}(x+xy^2)$

$\frac{\partial M}{\partial y} = 1+y^2$ $\qquad \frac{\partial N}{\partial x} = \frac{1}{4}(1+y^2)$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Now

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{(1+y^2)-\frac{1}{4}(1+y^2)}{\frac{1}{4}x(1+y^2)} = \frac{3/4}{1/4\, x} = \frac{3}{x}$$

$$I.F = e^{\int \frac{3}{x}dx} = e^{3\log x} = e^{\log x^3} = x^3$$

Multiplying (1) by $x^3$

$$\left(x^3y+\frac{x^3y^3}{3}+\frac{x^5}{2}\right)dx+\frac{1}{4}(x^4+x^4y^2)dy=0 \quad \text{——(2)}$$

clearly (2) is an exact.

Integrating, we get

$$\boxed{\frac{x^4y}{4}+\frac{x^4y^3}{12}+\frac{x^6}{12}=C}$$

3. $(x^2+y^2+1)dx-2xy\,dy=0$

4. $(3xy-2ay^2)dx+(x^2-2axy)dy=0$

5. $(x^2+y^2+x)dx+xy\,dy=0$

6. $(2y^3+2)dx+3xy^2dy=0.$

Method 4: If $Mdx+Ndy=0$ such that $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=f(y)$ (or) $k$(const)

then $I.F. = e^{\int f(y)dy}$ or $e^{\int k dy}$.

1. $(xy^3+y)dx+2(x^2y^2+x+y^4)dy=0$

2. $(y^4+2y)dx+(xy^3+2y^4-4x)dy=0$

3. $(xy^2-x^2)dx+(3x^2y^2+x^2y-2x^3+y^2)dy=0$

4. $(y+y^2)dx+xy\,dy=0.$

Method 5: If $Mdx+Ndy=0$ can be put in the form of $x^{\alpha}y^{\beta}(my\,dx+nx\,dy)+x^{\alpha'}y^{\beta'}(m'y\,dx+n'x\,dy)=0$ where $\alpha, \beta; \alpha', \beta'; m', n'; m, n$ are constants then the given eqn has an I.F. $x^h y^k$ where $h$ & $k$ must be obtained by applying the condition that the given eqn must become exact after multiplying by $x^h y^k$...

i.e. $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ $\qquad x^a y^b(my\,dx+nx\,dy)+x^c y^d(py\,dx+qx\,dy)=0$ $(mp-nq\neq 0)$

$h, k \rightarrow \frac{a+h+1}{m}=\frac{b+k+1}{n}$ ; $\frac{c+h+1}{p}=\frac{d+k+1}{q}$

# Linear Differential Equations

The first order diff. eqn of the form $\frac{dy}{dx} + P(x) \cdot y = Q(x)$ where $P(x)$ & $Q(x)$ are functions of $x$ only (or) constants (OR) $\frac{dx}{dy} + P(y) \cdot x = Q(y)$ where $P(y)$ & $Q(y)$ are functions of $y$ only (or) constants. is called a linear diff. eqn.

Note: [1] There are two types of linear diff. eqns.

[2]. To solve linear diff eqns, we take a factor is called "Integrating factor".

## working rule:

Type 1: (i) $\frac{dy}{dx} + P(x) y = Q(x)$

(ii) Find $I.F = e^{\int P dx}$

(iii) G.S is $y(I.F) = \int Q (I.F) \cdot dx + C$

Type 2: (i) $\frac{dx}{dy} + P(y) x = Q(y)$

(ii) Find I.F $= e^{\int P dy}$

(iii) G.S is $x (I.F) = \int Q (I.F) dy + C.$

Problems:

[1] Find I.F. of $\sin x \frac{dy}{dx} + 3y = \cos x$ — (1)

$\Rightarrow \frac{dy}{dx} + (3 \operatorname{cosec} x) y = \cot x.$

$$I.F = e^{\int 3 \operatorname{cosec} x \, dx} = e^{3 \log \tan(x/2)}$$

$$= \tan^3(x/2).$$

[2]. Solve $\frac{dy}{dx} + y \cot x = 2 \cos x.$

$$I.F = e^{\int \cot x} = e^{\log \sin x}.$$

$\therefore$ G.S. is $y \sin x = \int 2 \cos x \cdot \sin x \, dx + C$

$$= \int \sin 2x \, dx + C$$

$$\boxed{y \sin x = -\frac{\cos 2x}{2} + C.}$$

3. $\cos^2 x \frac{dy}{dx} + y = \tan x$

$\Rightarrow \frac{dy}{dx} + \sec^2 x\, y = \tan x \sec^2 x.$

$I.F = e^{\int \sec^2 x\, dx}$

$= e^{\tan x}.$

$\therefore$ G.S is

$y\, e^{\tan x} = \int \tan x \sec^2 x\, e^{\tan x} dx + c$

$= \int t e^t dt + c$ $\quad\Big\|$ $\tan x = t$ $\Rightarrow \sec^2 x\, dx = dt$

$= e^t(t-1) + c$

$= e^{\tan x}(\tan x - 1) + c.$

$$\boxed{y e^{\tan x} = e^{\tan x}(\tan x - 1) + c}$$

4. $\frac{dy}{dx} + \frac{y}{x} = x^n$

$I.F = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

G.S is

$y x = \int x^{n+1} dx + c$

$= \frac{x^{n+2}}{n+2} + c$

5. $x\frac{dy}{dx} - 2y = x^2$

$\Rightarrow \frac{dy}{dx} - \frac{2}{x} y = x$

$I.F = e^{-\int \frac{2}{x} dx} = e^{-2\log x} = \frac{1}{x^2}$

G.S is $y \frac{1}{x^2} = \int \frac{1}{x^2} \cdot x\, dx + c$

$= \int \frac{1}{x} dx + c$

$$\boxed{y/x^2 = \log x + c}$$

6. $\frac{dy}{dx} + \frac{2y}{x} = \sin x.$

$I.F = e^{\int \frac{2}{x} dx} = e^{2\log x} = x^2$

G.S is $y x^2 = \int x^2 \sin x \, dx + c$

$= -x^2 \cos x + \int 2x \cos x \, dx + c$

$= -x^2 \cos x + 2[x \sin x + \cos x] + c$

[7]. $x \log x \frac{dy}{dx} + y = 2\log x.$

$\Rightarrow \frac{dy}{dx} + \frac{1}{x \log x} y = \frac{2}{x}$

$I.F = e^{\int \frac{1}{x\log x} dx} = e^{\log(\log x)} = \log x.$

G.S is $y \log x = \int \frac{2}{x} \log x \, dx + c$

$= 2\frac{(\log x)^2}{2} + c$

$= (\log x)^2 + c.$

H.W. [8]. $x\frac{dy}{dx} + 2y = x^2 \log x.$

[9]. $(\sin 2x)\frac{dy}{dx} = y + \tan x.$

$\Rightarrow \frac{dy}{dx} - (\operatorname{cosec} 2x) y = \frac{1}{2}\sec^2 x$

$I.F = e^{-\int \operatorname{cosec} 2x \, dx} = e^{-\frac{\log(\tan x)}{2}}$

$= e^{\log \frac{1}{\sqrt{\tan x}}}$

$= \frac{1}{\sqrt{\tan x}}.$

G.S is

$y \frac{1}{\sqrt{\tan x}} = \int \frac{1}{2}\sec^2 x \frac{1}{\sqrt{\tan x}} dx + c$

$= \frac{1}{2}\int \frac{1}{\sqrt{t}} dt + c$ $\quad \tan x = t$, $\sec^2 x\, dx = dt$

$= \frac{1}{2}\left(\frac{t^{1/2}}{1/2}\right) + c$

$= \sqrt{\tan x} + c.$

H.W. [10] $(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}.$

11. $\frac{dy}{dx} + \frac{y}{(1-x^2)^{3/2}} = \frac{x-\sqrt{1-x^2}}{(1-x^2)^2}$

$I.F = e^{\int \frac{1}{(1-x^2)^{3/2}} dx}$ let $x = \sin\theta$, $dx = \cos\theta\, d\theta$

$= e^{\int \sec^2\theta\, d\theta}$

$= e^{\tan\theta} = e^{\frac{x}{\sqrt{1-x^2}}}$

G.S is $y e^{\frac{x}{\sqrt{1-x^2}}} = \int \frac{x-\sqrt{1-x^2}}{(1-x^2)^2} e^{\frac{x}{\sqrt{1-x^2}}} dx + C$

$= \int \frac{\sin\theta - \cos\theta}{\cos^4\theta} e^{\tan\theta} \cos\theta\, d\theta + C.$

$= \int e^{\tan\theta} (\sec^2\theta \tan\theta + \sec^2\theta) d\theta$

$= \int [t e^t dt + e^t] dt$ where $\tan\theta = t$

$= e^t(t-1) + e^t + C$

$= e^{\tan\theta}(\tan\theta - 1) + e^{\tan\theta} + C$

12. $x\cos x \frac{dy}{dx} + (x\sin x + \cos x) y = 1$

$\Rightarrow \frac{dy}{dx} + \left(\tan x + \frac{1}{x}\right) y = \frac{\sec x}{x}$

$I.F = e^{\int(\tan x + \frac{1}{x}) dx} = e^{\log|\sec x| + \log x}$

$= x \sec x.$

G.S is

$y x \sec x = \int \sec^2 x\, dx + C$

$= \tan x + C$

13. $(x + 2y^3) \frac{dy}{dx} = y$

$y \frac{dx}{dy} = x + 2y^3$

$\Rightarrow \frac{dx}{dy} - \left(\frac{1}{y}\right) x = 2y^2$

$$I.F = e^{-\int \frac{1}{y} dy} = e^{-\log y} = 1/y$$

$\therefore$ G.S is $\frac{x}{y} = \int 2y^2 \cdot \frac{1}{y} dy + C$

$$= y^2 + C$$

[14]. $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

$\Rightarrow \frac{dx}{dy} + \frac{1}{(1+y^2)} x = \frac{\tan^{-1}y}{1+y^2}$

$I.F = e^{\tan^{-1}y}$

G.S is $x e^{\tan^{-1}y} = \int e^{\tan^{-1}y} \frac{\tan^{-1}y}{1+y^2} dy + C$

$$= e^{\tan^{-1}y}(\tan^{-1}y - 1) + C$$

[15]. $y^2 + \left(x - \frac{1}{y}\right)\frac{dy}{dx} = 0$

$\Rightarrow y^2 \frac{dx}{dy} + x = \frac{1}{y}$

$\Rightarrow \frac{dx}{dy} + \frac{1}{y^2} x = \frac{1}{y^3}$

$I.F = e^{\int \frac{1}{y^2} dy} = e^{-1/y}$

G.S is $e^{-1/y} \cdot x = \int \frac{1}{y^3} e^{-1/y} dy + C$

$$= e^{-1/y}\left[\frac{1}{y} - 1\right] + C$$

Let $\frac{1}{y} = t$

$-\frac{1}{y^2} dy = dt$

$\Rightarrow \frac{1}{y^2} dy = -dt$

[16]. $(x+y+1)\frac{dy}{dx} = 1$

$\Rightarrow \frac{dx}{dy} = x + y + 1$

$\Rightarrow \frac{dx}{dy} - x = y + 1$

$I.F = e^{-\int dy} = e^{-y}$

G.S is $xe^{-y} = \int (y+1) e^{-y} dy + C$

$= \int y e^{-y} dy + \int e^{-y} dy + C$

$= + \int t e^{t} dt + \int e^{t}(-dt) + C$ $\quad\Big\|\quad -y = t,\ dy = -dt$

$$= +e^{t}(t-1) - e^{t} + c$$

$$= +e^{-y}(-y-1) - e^{-y} + c$$

$$= -e^{-y}(y+2) + c$$

17. $(1+x+xy^2)\frac{dy}{dx} + (y+y^3) = 0$

18. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 1 = 0$ —— (1)

putting $\frac{dy}{dx} = p$

$\Rightarrow x\frac{dp}{dx} + p + 1 = 0$ —— (2)

$\Rightarrow \frac{dp}{dx} + \left(\frac{1}{x}\right)p = -\frac{1}{x}$

I.F $= e^{\int \frac{1}{x}dx} = x.$

G.S of (1) is

$$px = \int -\frac{1}{x}\cdot x\,dx + c$$

$$xp = -x + c$$

$\Rightarrow x\frac{dy}{dx} = -x + c$

$\Rightarrow dy = \left(-1 + \frac{c}{x}\right)dx$

$\Rightarrow y = -x + c\log x + c'$

$\Rightarrow \boxed{x + y = c\log x + c'}$

## Equations reducible to linear form:

I. $f'(y)\frac{dy}{dx} + P f(y) = Q$

where P & Q are functions of x only.

putting $f(y) = v$

$\Rightarrow f'(y)\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx} + Pv = Q$

Clearly which is linear.

II. $f'(x)\frac{dx}{dy} + P f(x) = Q$:

where P & Q are functions of y only.

putting $f(x) = v$

$\Rightarrow f'(x)\frac{dx}{dy} = \frac{dv}{dy}$

$\therefore \frac{dv}{dy} + Pv = Q$.

which is linear.

(III) Bernolli's equation.

An equation of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$ where P & Q are functions of x alone (or) constants and 'n' is constant such that $n \neq 0$ & $n \neq 1$; is called Bernoulli's diff. equation.

(or)

$\frac{dx}{dy} + P(y)x = Q(y)x^n$

where P & Q are functions of y alone. (or) constants and 'n' is constant such that $n \neq 0$ & $n \neq 1$; is called Bernoulli's diff. eqn.

working rule:

$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$ —— (1)

put $y^{1-n} = z$

$y^{-n}(1-n)\frac{dy}{dx} = \frac{dz}{dx}$.

$$\Rightarrow y^{-n}\frac{dy}{dx} = \left(\frac{1}{1-n}\right)\frac{dz}{dx}.$$

$$(1) \equiv \frac{dz}{dx} + (1-n)\,P(x)\,z = Q(x).$$

clearly which is linear.

**Solve the following diff. equs.**

(1) $\sec^2 y \frac{dy}{dx} + 2x\tan y = x^3$ — (1)

put $\tan y = t$.

$$\sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$$

$$(1) \equiv \frac{dt}{dx} + 2xt = x^3 \quad \text{— (2)}$$

$$I.F. = e^{\int 2x\,dx} = e^{x^2}.$$

G.S. of (1) is

$$t \cdot e^{x^2} = \int x^3 e^{x^2}\,dx + c$$

$$= \frac{1}{2}\int z e^{z}\,dz + c$$

$$= \frac{1}{2} e^{z}(z-1) + c$$

let $x^2 = z$

$2x\,dx = dz$.

$x\,dx = \frac{1}{2}dz$.

$$\tan y \cdot e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + c$$

[2]. $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$

Dividing by $e^y$ we get

$$e^{-y}\frac{dy}{dx} + \frac{1}{x}e^{-y} = \frac{1}{x^2} \quad \text{— (1)}$$

But $e^{-y} = t \Rightarrow e^{-y}\frac{dy}{dx} = -\frac{dt}{dx}$

$$\frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x^2} \quad \text{— (2)}$$

$$I.F. = \frac{1}{x}$$

G.S is

$$t\,\frac{1}{x} = \int\left(-\frac{1}{x^2}\right)\frac{1}{x}\,dx + c$$

$$= \frac{1}{2x^2} + c$$

$$\Rightarrow \frac{e^{-y}}{x} = \frac{1}{2x^2} + c$$

3. $x\frac{dy}{dx}+y=y^2\log x.$

4. $\frac{dy}{dx}+\frac{xy}{1-x^2}=x\sqrt{y}$

$y^{-1/2}\frac{dy}{dx}+\frac{x}{1-x^2}y^{1/2}=x$ — ①

put $y^{1/2}=t \Rightarrow \frac{1}{2}y^{-1/2}\frac{dy}{dx}=\frac{dt}{dx}$

$\Rightarrow \boxed{y^{-1/2}\frac{dy}{dx}=2\frac{dt}{dx}}$

① ≡ $\frac{dt}{dx}+\frac{x}{2(1-x^2)}t=\frac{x}{2}$

$I.F = e^{\int\frac{x}{2(1-x^2)}dx} = e^{-\frac{1}{4}\int\frac{-2x}{1-x^2}dx}$

$= e^{-\frac{1}{4}\log(1-x^2)}$

$= e^{-\frac{1}{4}\log(1-x^2)}$

$= \frac{1}{(1-x^2)^{1/4}}$

∴ G.S of ① is

$t\frac{1}{(1-x^2)^{1/4}}=\int\frac{1}{(1-x^2)^{1/4}}\left(\frac{x}{2}\right)dx+C$

$= -\frac{1}{4}\int\frac{-2x}{(1-x^2)^{1/4}}dx+C$

$= -\frac{1}{4}\frac{(1-x^2)^{3/4}}{3/4}+C$

$y^{1/2}(1-x^2)^{-1/4} = -\frac{1}{3}\frac{1}{(1-x^2)^{-3/4}}+C$

6. $\frac{dy}{dx}=e^{x-y}(e^x-e^y)$

$=e^{2x}e^{-y}-e^x$

$\Rightarrow \frac{dy}{dx}+e^x=e^{2x}e^{-y}$

$\Rightarrow e^y\frac{dy}{dx}+e^xe^y=e^{2x}$ — ①

put $e^y=t \Rightarrow e^y\frac{dy}{dx}=\frac{dt}{dx}$

$\frac{dt}{dx}+e^xt=e^{2x}$

$I.F = e^{\int e^x dx}=e^{e^x}.$

G.S is

$$te^{e^y} = \int e^{2x} e^{e^x} dx + c$$

$$= \int z e^{z} dz + c \qquad e^x = z, \quad e^x dx = dz.$$

$$= e^{z}(z-1) + c$$

$$e^y e^{e^x} = e^{e^x}(e^x - 1) + c$$

[7]. $\frac{dy}{dx}(x^2y^3 + xy) = 1$

$$\Rightarrow \frac{dx}{dy} = x^2y^3 + xy$$

$$\Rightarrow \frac{dx}{dy} - yx = x^2y^3$$

$$\Rightarrow x^{-2}\frac{dx}{dy} - yx^{-1} = y^3 \quad \text{—(1)}$$

put $x^{-1} = t$

$$-x^{-2}\frac{dx}{dy} = \frac{dt}{dy}$$

(1) ≡ $-\frac{dt}{dy} - yt = y^3$

$$\Rightarrow \frac{dt}{dy} + yt = -y^3$$

$$IF = e^{\int y\,dy} = e^{y^2/2}$$

G.S. is

$$te^{y^2/2} = -\int y^3 e^{y^2/2} dy + c$$

$$= -\int 2z e^{z} dz + c \qquad y^2/2 = z, \quad y^2 = 2z, \quad 2y\,dy = 2dz, \quad y\,dy = dz$$

$$= -2e^{z}(z-1) + c$$

$$x^{-1}e^{y^2/2} = -2e^{y^2/2}(y^2/2 - 1) + c$$

[8]. $\frac{dy}{dx} + \frac{y}{x} = y^2$

[9]. $\frac{dy}{dx} = x^3y^3 - xy$

[10]. $\frac{dy}{dx} + 1 = e^{x-y}$

[11]. $xy^2\frac{dy}{dx} - 2y^3 = 2x^3$

[12]. $x\frac{dy}{dx} + y\log y = xye^x$.

13. $2y\cos y^2 \frac{dy}{dx} - \frac{2\sin y^2}{x+1} = (x+1)^3$ — ①

put $\sin y^2 = t$

$2y\cos y^2 \frac{dy}{dx} = \frac{dt}{dx}$

① ≡ $\frac{dt}{dx} - \frac{2t}{x+1} = (x+1)^3$

I.F $= e^{-\int \frac{2}{x+1}dx} = e^{-2\log(x+1)} = \frac{1}{(x+1)^2}$

G.S. $t\frac{1}{(x+1)^2} = \int (x+1)dx + C$

$$\boxed{\sin y^2 \frac{1}{(x+1)^2} = \frac{x^2}{2} + x + C}$$

14. $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

$\Rightarrow (\cos^2 y)^{-1}\frac{dy}{dx} + \frac{x \cdot 2\sin y\cos y}{\cos^2 y} = x^3$

$\Rightarrow (\cos^2 y)^{-1}\frac{dy}{dx} + 2x\tan y = x^3$ — ①

put $\tan y = t$

$\sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$

① ≡ $\frac{dt}{dx} + 2xt = x^3$

I.F $= e^{\int 2x\,dx} = e^{x^2}$

$te^{x^2} = \int e^{x^2} x^3 dx + C$

$= \frac{1}{2}\int e^{z} z\, dz + C$

$= \frac{1}{2}e^{z}(z-1) + C$

$x^2 = z$

$2x\,dx = dz$

$x\,dx = \frac{dz}{2}$

$\tan y\, e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + C$

15. Find the eqn of the curve which passes through the point $(1,-1)$ and satisfies the diff. eqn

H.W.

$x\frac{dy}{dx} + y = x^2y^2$.

2001 → $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x^2}(\log z)^2$ —— ①

Dividing by $z(\log z)^2$, we get

$$\frac{1}{z(\log z)^2}\cdot\frac{dz}{dx} + \frac{1}{x}(\log z)^{-1} = \frac{1}{x^2} \quad \text{—— ②}$$

put $(\log z)^{-1} = t$

$$\frac{(-1)(\log z)^{-2}}{z}\frac{dz}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{1}{z(\log z)^2}\frac{dz}{dx} = -\frac{dt}{dx}$$

② ≡ $-\frac{dt}{dx} + \frac{1}{x}t = \frac{1}{x^2}$

$$\Rightarrow \frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x^2} \quad \text{—— ③}$$

$$I.F. = e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

∴ G.S of ③ is

$$\frac{1}{x}t = \int -\frac{1}{x^2}\cdot\frac{1}{x}dx + c$$

$$= -\int x^{-3}dx + c$$

$$= -\frac{x^{-2}}{-2} + c$$

$$= \frac{1}{2x^2} + c$$

$$\therefore \boxed{\frac{1}{x}(\log z)^{-1} = \frac{1}{2x^2} + c}$$

Miscellaneous

34(i)

To find the N.C and S.C that the equation $M\,dx + N\,dy = 0$ may be exact.

Proof Part 1: Let $M\,dx + N\,dy = 0$ be an exact. —— (1)

then by defn, $M\,dx + N\,dy = du$

where $u$ is a function of $x$ & $y$.

and $M\,dx + N\,dy = du$

$$= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\Rightarrow M = \frac{\partial u}{\partial x} \quad \Big| \quad N = \frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \; ; \; \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\left(\because \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}\right)$.

Part 2: Let $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

To P.T $M\,dx + N\,dy = 0$ is exact.

Let $\int M\,dx = u$ —— (2)

where integration has been performed by treating $y$ as constant.

$$\therefore \frac{\partial}{\partial x}\left[\int M\,dx\right] = \frac{\partial u}{\partial x}.$$

$$\Rightarrow M = \frac{\partial u}{\partial x}. \quad \text{—— (3)}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}. \quad \text{—— (4)}$$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$.

(4) $\equiv \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$.

integrating both sides w.r.t $x$ by treating $y$ as const.

$\therefore N = \frac{\partial u}{\partial y} +$ a function of $y$.

$= \frac{\partial u}{\partial y} + f(y)$ say, —— (5)

from (3) & (5), we have

$Mdx + Ndy \equiv \frac{\partial u}{\partial x} dx + \left[\frac{\partial u}{\partial y} + f(y)\right] dy$

$= \left[\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right] + f(y)\, dy$

$= du + f(y)\, dy$

$= d\left[u + \int f(y)\, dy\right]$. —— (6)

which is an exact diff.

Hence $Mdx + Ndy = 0$ is an exact diff. equation.

Cor: solution of an exact diff. eq.

If the equation $Mdx + Ndy = 0$ is exact.

then $Mdx + Ndy = d\left[u + \int f(y) dy\right]$

$\Rightarrow Mdx + Ndy = 0 \Rightarrow d\left[u + \int f(y) dy\right] = 0$ (by using (6))

integrating both sides, we get

$u + \int f(y)\, dy = c$.

(7) $\equiv u = \int_{y\text{-const}} M dx$

(5) $\equiv f(y) =$ terms in $N$ not containing $x$

$\therefore \int M dx + \int (\text{terms in } N \text{ containing } x)\, dy = c$.

## Proofs

**Method 1:-** If $Mdx+Ndy=0$ is a homogeneous and $Mx+Ny\neq 0$ then $\frac{1}{Mx+Ny}$ is an I.F.

**Proof.** Given that $Mdx+Ndy=0$ —— (1)

where $M$ & $N$ are homogeneous functions of the same degree in $x$ & $y$.

By Euler's theorem on partial differentiation

$$\left.\begin{aligned} x\frac{\partial M}{\partial x}+y\frac{\partial M}{\partial y} &= nM \ \& \\ x\frac{\partial N}{\partial x}+y\frac{\partial N}{\partial y} &= nN \end{aligned}\right\} \text{—— (2)}$$

Now

$$(1)\times\frac{1}{Mx+Ny} \equiv \frac{M}{Mx+Ny}dx+\frac{N}{Mx+Ny}dy=0. \text{ —— (3)}$$

It will be exact if

$$\frac{\partial}{\partial y}\left(\frac{M}{Mx+Ny}\right)=\frac{\partial}{\partial x}\left(\frac{N}{Mx+Ny}\right)$$

i.e
$$\frac{(Mx+Ny)\frac{\partial M}{\partial y}-M\left(x\frac{\partial M}{\partial y}+y\frac{\partial N}{\partial y}+N\right)}{(Mx+Ny)^2}$$

$$=\frac{(Mx+Ny)\frac{\partial N}{\partial x}-N\left(M+x\frac{\partial M}{\partial x}+y\frac{\partial N}{\partial x}\right)}{(Mx+Ny)^2}$$

i.e $\cancel{Mx\frac{\partial M}{\partial y}}+Ny\frac{\partial M}{\partial y}-\cancel{Mx\frac{\partial M}{\partial y}}-My\frac{\partial N}{\partial y}-\cancel{MN}$

$=Mx\frac{\partial N}{\partial x}-Ny\frac{\partial N}{\partial x}-\cancel{MN}-Mx$

$$\Rightarrow N\left(x\frac{\partial M}{\partial x}+\frac{\partial M}{\partial y}y\right)=M\left(x\frac{\partial N}{\partial x}+y\frac{\partial N}{\partial y}\right)$$

$\Rightarrow N \cdot nM = M \cdot nN$ (by using ②).

which is true. Hence the result.

Alternate Method :—

The given equation is $Mdx + Ndy = 0$ — ①

where M & N are homogeneous functions of the same degree in $x$ & $y$.

Now we have

$\cancel{Mdx + Ndy = \frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right)\right]}$

$$Mdx + Ndy \equiv \frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx-Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$$

$$\Rightarrow Mdx + Ndy \equiv \frac{1}{2}\left[(Mx+Ny)\,d(\log xy) + (Mx-Ny)\,d\left(\log\frac{x}{y}\right)\right]$$

Now dividing by $Mx+Ny$ (which is $\neq 0$)

$$\frac{Mdx+Ndy}{Mx+Ny} \equiv \frac{1}{2}\left[d(\log(xy)) + \frac{Mx-Ny}{Mx+Ny}\,d\left(\log\frac{x}{y}\right)\right]$$

Since M & N are homogeneous functions of the same degree in $x$ & $y$,

the expression $\frac{Mx-Ny}{Mx+Ny}$ is homogeneous and equal to a function of $\frac{x}{y}$ say $f\left(\frac{x}{y}\right)$

$$\therefore \frac{Mdx+Ndy}{Mx+Ny} \equiv \frac{1}{2}d(\log xy) + \frac{1}{2}f\left(\frac{x}{y}\right)d\left(\log\frac{x}{y}\right)$$

$$\Rightarrow \frac{x}{y} = e^{\log\left(\frac{x}{y}\right)},$$

$$\Rightarrow f\left(\frac{x}{y}\right) = f\left(e^{\log\frac{x}{y}}\right) = F\left(\log\frac{x}{y}\right)$$

$$\Rightarrow \frac{M\,dx+N\,dy}{Mx+Ny} \equiv \frac{1}{2}\, d(\log xy) + \frac{1}{2} F\left(\log\frac{x}{y}\right) d\left(\log\frac{x}{y}\right)$$

which is an exact differential.

$$\Rightarrow \frac{M\,dx+N\,dy}{Mx+Ny} = 0$$

$$\Rightarrow \frac{M}{Mx+Ny}\,dx + \frac{N}{Mx+Ny}\,dy = 0$$

is an exact diff. eqn.

---

Method 2: If the equation $M\,dx+N\,dy=0$ is of the form $y f_1(xy)\,dx + x f_2(xy)\,dy = 0$ then $\frac{1}{Mx-Ny}$ is an integrating factor (I.F.), provided $Mx-Ny \neq 0$.

proof The given equation is $M\,dx+N\,dy=0$ — (1)

where $M = y f_1(xy)$; $N = x f_2(xy)$.

Now we have

$$M\,dx+N\,dy = \frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right) + (Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$$

$$\Rightarrow M\,dx+N\,dy \equiv \frac{1}{2}\left[(Mx+Ny)\,d(\log xy) + (Mx-Ny)\,d\left(\log\frac{x}{y}\right)\right]$$

Dividing by $Mx-Ny$; $(Mx-Ny\neq 0)$ we get,

$$\frac{M\,dx+N\,dy}{Mx-Ny} \equiv \frac{1}{2}\left[\frac{Mx+Ny}{Mx-Ny}\,d(\log(xy)) + d\left(\log\left(\frac{x}{y}\right)\right)\right].$$

$$= \frac{1}{2}\left[\frac{xy f_1(xy) + xy f_2(xy)}{xy f_1(xy) - xy f_2(xy)} d(\log(xy)) + d\left(\log\left(\frac{x}{y}\right)\right)\right]$$

$$= \frac{1}{2}\left[f_3(xy)\, d(\log(xy)) + d\left(\log \frac{x}{y}\right)\right]$$

Since $xy = e^{\log xy}$

$$\Rightarrow f(xy) = f\left(e^{\log xy}\right) = F(\log xy)$$

$$\therefore \frac{Mdx + Ndy}{Mx - Ny} = \frac{1}{2} F(\log xy)\, d(\log xy) + \frac{1}{2} d\left(\log \frac{x}{y}\right)$$

which is an exact diff.

$$\Rightarrow \frac{Mdx + Ndy}{Mx - Ny} = 0$$

$$\Rightarrow \frac{M}{Mx - Ny} dx + \frac{N}{Mx - Ny} dy = 0$$ is an

exact diff. eqn. ~~exact diff~~.

$\therefore \frac{1}{Mx - Ny}$ is an integrating factor.

**Method 3:** If in the equation $Mdx + Ndy = 0$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of $x$ only

$= f(x)$ say ~~then~~ (or) $K$ (constant) then $e^{\int f(x) dx}$ is an integrating factor.

# Set - II

## Linear Equations with constant coefficients.

A linear differential eqn of order 'n' of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = Q \quad \text{--- (1)}$$

where $a_1, a_2, \ldots a_{n-1}, a_n$ are all constants and Q is any function 'x' is called a linear diff. eqn with constant coefficients.

For our convenience, the operators $\frac{d}{dx}, \frac{d^2}{dx^2}, \ldots \frac{d^{n-1}}{dx^{n-1}}, \frac{d^n}{dx^n}$ are also denoted by $D, D^2, D^3, \ldots D^n$ respectively.

∴ The equation (1) can be written as

$$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots + a_{n-1} Dy + a_n y = Q$$

$$\Rightarrow [D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n] y = Q$$

$$\Rightarrow f(D)\, y = Q \quad \text{--- (2)} \quad \text{where } f(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n.$$

**Homogeneous eqn**: If $Q=0$ then (2) is called homogeneous eqn with constant coefficients.

i.e, a linear homogeneous eqn of order 'n' is

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n) y = 0 \quad \text{--- (3)}$$

→ If $y = f(x)$ is the general solution of (3) and $y = P(x)$ is any particular solution of the eqn (2) is not containing any arbitrary constant. Then $y = f(x) + P(x)$ is called the g.s. of (2)

**The method of solving a linear eqn is dividing into two parts:**

→ first we find the general solution of the eqn (3). It is called the Complementary function (C.F).

It must be contain many arbitrary constants as is the order of the given diff-eqn.

→ Next, we find a particular solution of ② which doesnot contain arbitrary constants this is called the particular Integral (P.I).

→ If we add (CF) and (P.I) then we get the general solution of (2)

i.e, The general solution of ② is

$$y = C.F + P.I \quad (or)$$

$$y = y_c + y_p.$$

Auxilary Eqn (A.E): Now we consider the diff. eqn $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n) y = 0$ — ①

i.e, $f(D)y = 0$

where $f(D) = D^n + a_1 D^{n-1} + \dots + a_{n-1}D + a_n$.

The eqn $f(m) = 0$ is called the A.E of ① where $m = D$.

∴ A.E of ① is given by

$$m^n + a_1 m^{n-1} + \dots + a_{n-1}m + a_n = 0$$

Clearly it will have 'n' roots.

These roots may be real (or) complex or surds.

To find the C.F of $f(D)y = 0$:-

Consider the eqn $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n) y = 0$

i.e, $f(D)y = 0$

The A.E. of ① is $f(m) = 0$

i.e, $m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n = 0$ — ②

Case(i): when all the roots of ② are real and distinct

Let $m = m_1, m_2, \dots m_n$ be the 'n' real and distinct roots of ②.

Then $y=e^{m_1x}, y=e^{m_2x}, \ldots\ldots y=e^{m_nx}$ are independent solutions of (1).

Hence the g.s of (1) is

$$y = C_1e^{m_1x}+C_2e^{m_2x}+\ldots\ldots+C_ne^{m_nx}.$$

where $C_1, C_2, \ldots\ldots C_n$ are constants.

Ex: $(D-m_1)y=0$ — (i)

$\Rightarrow Dy-m_1y=0$

$\Rightarrow \frac{dy}{dx} = m_1y$

$\Rightarrow \frac{dy}{y} = m_1 dx$

$\Rightarrow \log y = m_1x+\log c$

$\Rightarrow \log(y/c) = m_1x$

$\Rightarrow y/c = e^{m_1x}$

$\Rightarrow \boxed{y = ce^{m_1x}.}$

Case(ii): When two roots of (2) are equal and other roots are distinct.

Let $m_1=m_2$ i.e, $m_1, m_1; m_3, m_4, \ldots\ldots m_{n-1}, m_n$ be the real and distinct roots of (2).

Then g.s of (1) is

$$y=(C_1+C_2x)e^{m_1x}+C_3e^{m_3x}+\ldots\ldots+C_ne^{m_nx}.$$

Ex: $(D-m_1)^2y=0$ — (1) in which the roots are equal.

$\Rightarrow (D-m_1)(D-m_1)y=0$

$\Rightarrow (D-m_1)v=0$ where $v=(D-m_1)y$ — (2)

$\Rightarrow v = C_1e^{m_1x}$

(2) $\equiv (D-m_1)y = C_1e^{m_1x}$

$\Rightarrow Dy-m_1y = C_1e^{m_1x}$

$\Rightarrow \frac{dy}{dx}-m_1y = C_1e^{m_1x}$ — (3)

$\therefore \boxed{I.F = e^{-m_1x}}$

$\therefore$ G.S is $ye^{-m_1x} = \int C_1dx+C_2 = C_1x+C_2$

$\therefore \boxed{y=(C_1x+C_2)e^{m_1x}}$

Case(iii) when three roots are equal.

$\therefore$ G.S of (1) is

$$y=(C_1+C_2x+C_3x^2)e^{m_1x}+C_4e^{m_4x}+\cdots+C_ne^{m_nx}.$$

Case(iv): When all the roots are equal.

$\therefore$ G.S. of (1) is

$$y=(C_1+C_2x+C_3x^2+\cdots+C_{n-1}x^{n-2}+C_nx^{n-1})e^{m_1x}$$

Case(v): When the A.E. of (1) has $\alpha\pm i\beta$ as a pair of complex roots.

Let $m_1=\alpha+i\beta$ & $m_2=\alpha-i\beta$

$\therefore$ G.S. of (1) is

$$y=C_1e^{(\alpha+i\beta)x}+C_2e^{(\alpha-i\beta)x}$$

$$=C_1e^{\alpha x}e^{i\beta x}+C_2e^{\alpha x}e^{-i\beta x}$$

$$=e^{\alpha x}\left[C_1(\cos\beta x+i\sin\beta x)+C_2(\cos\beta x-i\sin\beta x)\right]$$

$$=e^{\alpha x}\left[(C_1+C_2)\cos\beta x+i(C_1-C_2)\sin\beta x\right]$$

$$=e^{\alpha x}\left[A\cos\beta x+B\sin\beta x\right]$$

where $A=C_1+C_2$ ; $B=i(C_1-C_2)$; $i=\sqrt{-1}$

$\rightarrow$ If the imaginary roots are repeated, say $\alpha+i\beta$ & $\alpha-i\beta$ occur twice then the solution will be

$$y=e^{\alpha x}\left[(A+Bx)\cos\beta x+(C+Dx)\sin\beta x\right]$$

Note: [1] The expression $e^{\alpha x}(A\cos\beta x+B\sin\beta x)$ can also be written as

$Ae^{\alpha x}\sin(\beta x+B)$ or $Ae^{\alpha x}\cos(\beta x+B)$.

(2) If AE of (1) has $(\alpha\pm\sqrt{\beta})$ a pair of roots then G.S of (1) is

$$y=e^{\alpha x}\left[C_1\cosh\sqrt{\beta}x+C_2\sinh\sqrt{\beta}x\right]$$

some times it may be written as

$$y=C_1e^{\alpha x}\cosh(\sqrt{\beta}x+C_2)$$

→ If the roots $(\alpha \pm \sqrt{\beta})$ is repeated then the G.S is $y = e^{\alpha x}\left[(c_1+c_2x)\cosh\sqrt{\beta}x + (c_3+c_4x)\sinh\sqrt{\beta}x\right]$

**Problems:**

→ find C.F. of $(D^2-3D+2)y=0$.

Sol^n: Given that $(D^2-3D+2)y=0$

$\Rightarrow f(D)y=0 \quad \text{---}(1)$ where $f(D)=D^2-3D+2$

A.E. of (1) is $f(D)=0$

i.e, $D^2-3D+2=0$

$\Rightarrow (D-1)(D-2)=0$

$\Rightarrow D=1,2$

$\therefore$ C.F. of (1) is $\boxed{y=c_1e^x+c_2e^{2x}}$

$y=e^x, y=e^{2x}$ are independent sol^n of (1)

$\therefore$ C.F of (1) is $y=c_1e^x+c_2e^{2x}$

→ Solve $\dfrac{d^3y}{dx^3}-4\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}-2y=0$

→ solve $(D^4-81)y=0$

→ solve $(D^4+D^2+1)y=0$

Sol^n: A.E. is $f(D)=0$

$\Rightarrow D^4+D^2+1=0$

$\Rightarrow (D^2+1)^2-D^2=0$

$\Rightarrow (D^2+D+1)(D^2+1-D)=0$

$\Rightarrow D^2-D+1=0, \quad D^2+D+1=0$

$D=\dfrac{1\pm\sqrt{1-4}}{2}; \quad D=\dfrac{-1\pm\sqrt{1-4}}{2}$

$D=\dfrac{1\pm\sqrt{3}i}{2}, \dfrac{-1\pm\sqrt{3}i}{2}$

→ Find c.f. of $(D^2+a^2)y=0$

→ $(D^3+6D^2+12D+8)y=0$

→ $(D^2+D+1)^2y=0$ (or) $(D^4+2D^3+3D^2+2D+1)y=0$

→ $(D^4-7D^3+18D^2-20D+8)y=0$

→ $(D^4-D^3-9D^2-11D-4)y=0$

→ $\left(\dfrac{d^3y}{dx^3}+y\right)=0$.

→ $(D^2-2D+5)y=0$ given that (i) $y=0$ when $x=0$
Ans:- $e^x(2\sin x)$ (ii) $\frac{dy}{dx}=4$ when $x=0$.

→ $(D^2+4)y=0$ $(A\cos 2x+B\sin 2x)$

→ $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$ with $y=0$, $x=0$ and $\frac{dy}{dx}=0$. (0)

**To find the particular Integral:**

Let the given diff eqn be

$$(D^n+A_1D^{n-1}+\dots\dots+A_{n-1}D+A_n)y=Q$$

where $D=\frac{d}{dx}$.

$\Rightarrow f(D)y=Q$., where $f(D)=D^n+A_1D^{n-1}+\dots+A_{n-1}D+A_n$.

Its g.s. is $\boxed{y=CF+P.I}$

**Inverse operator:**

→ Since $f(D)\cdot\frac{1}{f(D)}Q=Q$.

$\Rightarrow \frac{1}{f(D)}$ is the inverse operator of $f(D)$

→ Since D is the diff operator. $\frac{1}{D}D=\int D$.

$\Rightarrow \frac{1}{D}$ is an integral operator of D.

**P.I of $f(D)y=Q$:**

Since $y=\frac{1}{f(D)}Q$ satisfies the eqn $f(D)y=Q$.

$\therefore$ P.I of $f(D)y=Q$ is $\frac{1}{f(D)}Q$.

**Methods for finding P.I:-**

Case(i): To find P.I when $Q=e^{ax}$ when $f(a)\neq 0$

Since $De^{ax}=ae^{ax}$; $D^2e^{ax}=a^2e^{ax}$, ..... $D^ne^{ax}=a^ne^{ax}$.

$\Rightarrow f(D)e^{ax}=f(a)e^{ax}$; $f(a)\neq 0$.

Now $e^{ax}=\frac{1}{f(D)}f(D)e^{ax}=\frac{1}{f(D)}\{f(a)e^{ax}\}$.

$$\Rightarrow \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad (\because f(a) \neq 0).$$

Working rule :-

If $f(D)y = e^{ax}$; $f(a) \neq 0$

then $P.I = \frac{1}{f(D)} e^{ax}$ where $Q = e^{ax}$.

$= \frac{1}{f(a)} e^{ax}$ (putting $D = a$ in $f(D)$).

Problems:

→ Find P.I of $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = e^{2x}$

Soln: $(D^2 - 7D + 12)y = e^{2x}$

$\Rightarrow f(D)y = Q$ where $f(D) = D^2 - 7D + 12$ and $Q = e^{2x}$.

$\therefore P.I = \frac{1}{f(D)} Q$

$= \frac{1}{D^2 - 7D + 12} e^{2x}$

$= \frac{1}{4 - 14 + 12} e^{2x} = \frac{1}{2} e^{2x}$.

→ Find P.I and solve $(D^2 + D + 1)y = e^{-x}$.

Soln Given that $(D^2 + D + 1)y = e^{-x}$.

$\Rightarrow f(D)y = e^{-x}$ —— ①

where $f(D) = D^2 + D + 1$.

A.E. of ① is $f(D) = 0$

$\Rightarrow D^2 + D + 1 = 0$

$\Rightarrow D = \frac{-1 \pm \sqrt{1-4}}{2}$

$= \frac{-1 \pm \sqrt{3}i}{2}$

$= -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$\therefore y_c = e^{-\frac{1}{2}x}\left(C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x\right)$

Now $P.I = \frac{1}{f(D)} Q = \frac{1}{D^2 + D + 1}(e^{-x})$

$= \frac{1}{1 - 1 + 1}(e^{-x}) = e^{-x}$.

$\therefore y_p = e^{-x}$.

$\therefore$ G.S. of ① is $y = y_c + y_p$.

$= e^{-\frac{1}{2}x}\left(C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x\right) + e^{-x}$.

→ solve $(D^2-2D-1)y = \cosh 3x$ $\quad \cosh x = \frac{e^{3x}+e^{-3x}}{2}$

→ solve $(D^2+4D+6)\, y = e^{2x}$

→ solve $\frac{d^2y}{dx^2} - 13\frac{dy}{dx} + 12y = 2008$ $\quad c_1e^{x} + c_2 e^{12x} + \frac{2008}{12}$

→ solve $(D^3-5D^2+7D-3)\,y = e^{2x}\cosh x$.

$= e^{2x}\,\frac{e^{x}+e^{-x}}{2}$

Case (ii) To find P.I when $Q = \sin(ax)$ or $\cos(ax)$ and $f(-a^2) \neq 0$

Since $D(\sin ax) = a\cos ax$ ; $D^2(\sin ax) = -a^2 \sin ax$

$D^3(\sin ax) = -a^3 \cos ax$ ; $D^4(\sin ax) = (-a^2)^2 \sin ax$

$(D^2)^2 \sin ax = (-a^2)^2 \sin ax.$

- - - - - - - - - - - - -

$(D^2)^n \sin ax = (-a^2)^n \sin ax.$

$\therefore f(D^2)\sin ax = f(-a^2) \sin ax$

where $f(-a^2) \neq 0$.

Now $\sin(ax) = \frac{1}{f(D^2)} f(D^2) \sin ax$

$= \frac{1}{f(D^2)} \{ f(-a^2) \sin ax \}$

$\Rightarrow \boxed{\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax.}$

Similarly $\boxed{\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax.}$

Working rule:-

To find P.I. $Q = \sin ax$ (or) $\cos ax$ and $f(-a^2) \neq 0$.

$P.I = \frac{1}{f(D^2)} \sin ax.$ $\quad$ (put $D^2 = -a^2$)

$= \frac{1}{f(-a^2)} \sin ax$

write $D^2 = -a^2$ ; $D^3 = -a^2 D$ , $D^4 = (-a^2)^2 = a^4$, - - - - -

Note: The linear factor $(D \pm \alpha)$ in the denominator may be removed by multiplying the Nr and Dr with $D \mp \alpha$ and then putting $D^2 = -a^2$

**Problems:**

→ solve $(D^2+4)y = \cos 4x$ $\left(A\cos 2x + B\sin 2x - \frac{1}{12}\cos 4x\right)$.

→ solve $(D^2-2D+5)y = \sin 3x$

→ solve $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \cos 2x$. $c_1e^x + (c_2+c_3x)e^{-x} + \frac{1}{25}(-2\sin 2x - \cos 2x)$

→ solve $(D^2-4D+3)y = \sin 3x \cos 2x$.

$2\sin A \sin B = \sin(A+B) + \sin(A-B)$

→ $(D^2-4)y = \sin^2 x$.

→ $(D^2-9)y = \cos^2 x$.

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 x = \frac{1+\cos 2x}{2}$

**Case (iii):** To find P.I. when $Q = x^m$ or polynomial of degree $m$ where $m$ is zero & +ve integer.

$$P.I = \frac{1}{f(D)} x^m \text{ (polynomial)}$$

Takeout common the lowest degree term from $f(D)$. The remaining factor in denominator is of the form $[1+F(D)]$ or $[1-F(D)]$ which is taken in the numerator with negative power.

Now expand $[1 \pm F(D)]^{-1}$ in ascending powers of $D$ by binomial theorem upto $D^m$ and operator upon $x^m$.

$$\left(\text{i.e, } \frac{1}{f(D)} x^m \Rightarrow \frac{1}{[1 \pm F(D)]} x^m \Rightarrow [1 \pm F(D)]^{-1} x^m.\right)$$

The following expansions by binomial theorem

(1) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$

(2) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

(3) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

(4) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

$$\left[\text{Since } (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots\right]$$

**Problems**

Solve the following diff. eqns.

[1] → $\frac{d^2y}{dx^2} - 4y = x^2$ $\quad c_1e^{2x} + c_2e^{-2x} - \frac{1}{4}\left(x^2 + \frac{1}{2}\right)$

[2] → $(D^2-3D+2)y = 2x^2$ $\quad c_1e^x + c_2e^{2x} + x^2 + 3x + \frac{7}{2}$

[3]. $(D^2+D+1)y = x^3$ $\quad e^{-x/2}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right] + x^3 - 3x^2 + 12$

[4]. $(D^2-1)y = 2+5x$. $\quad c_1e^x + c_2e^{-x} - 2 - 5x$

$x(c_1+c_2x+c_3x^2)$

5. $(D^3+8)y = x^4+2x+1$

~~6.~~ $(D^4-2D^3+D^2)y = x^3$

7. $(D^2+16)y = x^4+e^{3x}+\cos 3x$

8. $(D^2+D-2)y = x+\sin x$

9. $(D^2-5D+6)y = x+\sin 3x$

10. $(D^3-4D^2+5D)y-2=0$

$c_1e^{-x}+c_2e^{5x}+\frac{2x}{5}$

Case (iv): To find P.I when $Q = e^{ax}.V$

where $V=F(x)$

i.e $V = \sin ax$ (or) $\cos ax$ or $x^m$.

$$P.I = \frac{1}{f(D)}e^{ax}.V$$

Here take $e^{ax}$ outside after replacing $D$ by $(D+a)$ and operate $V$ by $\frac{1}{f(D+a)}$.

$$\therefore P.I = \frac{1}{f(D)}e^{ax}.V$$

$$= e^{ax}\frac{1}{f(D+a)}V.$$

Problems

→ Solve the following diff. equations.

1. $(D^2-4)y = x^2e^{3x}$ $\quad c_1e^{2x}+c_2e^{-2x}+\frac{1}{5}\left[x^2-\frac{12x}{5}+\frac{62}{25}\right].$

2. $(D^3-3D+2)y = x^2e^x$

3. $(D^2-2D+5)y = e^{2x}\sin x$

4. $(D^4-1)y = e^x\cos x$

5. $(D+1)^3y = x^2e^{-x}$

6. $D^2y = e^x\cos x$

7. $(D^2-4D+3)y = 2xe^{3x}+3e^x\cos 2x$

8. $(D^3-3D^2+4D-2)y = e^x+\cos x$

9. $(D^3-3D^2+3D-1)y = (x+1)e^x.$

Case (v): To find P.I when $Q = e^{ax}$ and $f(a) = 0$

$$P.I = \frac{1}{f(D)} e^{ax}$$

$$= \frac{1}{f(D)} e^{ax} \cdot 1$$

$$= e^{ax} \frac{1}{f(D+a)} \cdot 1$$

(or)

$$\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax} \quad \text{where } r = 1, 2, 3 \ldots$$

If $f(a) = 0$, then factorize $f(D)$, first operate on $e^{ax}$ by factor which doesnot vanish by putting 'a' for D and finally the other factor to apply the above formula).

Problems

find P.I of $[D^3 + D^2 - D - 1] y = e^x$

Sol

$$P.I = \frac{1}{D^3 + D^2 - D - 1} e^x$$

$$= \frac{1}{D^2(D+1) - (D+1)} e^x$$

$$= \frac{1}{(D+1)[D^2 - 1]} e^x$$

$$= \frac{1}{(D+1)^2 (D-1)} e^x$$

$$= \frac{1}{(D-1)} \left[ \frac{1}{(D+1)^2} e^x \right] = \frac{1}{D-1} \left[ \frac{1}{4} e^x \right]$$

$$= \frac{1}{4}\left[\frac{1}{D-1} e^{x}\right]$$

$$= \frac{1}{4}\left[\frac{x!}{1!} e^{x}\right]$$

$$= \frac{1}{4} x e^{x}.$$

→ find P.I of $(D^3+3D^2+3D+1)y = e^{-x}$

Sol. $P.I = \frac{1}{D^3+3D^2+3D+1} e^{-x} = \frac{1}{(D+1)^3} e^{-x}$

$$= e^{-x} \frac{1}{(D-1+1)^3}(1)$$

$$= e^{-x} \frac{1}{D^3}(1)$$

$$= e^{-x} \cdot \frac{1}{D^2}[x]$$

$$= e^{-x} \frac{1}{D}\left[\frac{x^2}{2}\right]$$

$$= e^{-x} \frac{x^3}{6}$$

(or)

$$P.I = \frac{1}{[D^3+3D^2+3D+1]} e^{-x} = \frac{x}{3D^2+6D+3}(e^{-x})$$

$$= \frac{x.x}{6D+6}(e^{-x})$$

$$= \frac{x.x.x}{6}(e^{-x})$$

$$= \frac{x^3}{6}(e^{-x}).$$

* Solve the following diff. eqns:

→ $(D+2)(D-1)^3 y = e^{-x}$

→ $(D^2+D-6)y = e^{2x}$

→ $(D^2-4D+4)y = e^{2x} + \sin 2x$

HW → $(D^2-3D+2)y = \cosh x$

HW → $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = e^x$

→ find P.I. of

sol $(D^2+a^2)y = \sin ax$

$$P.I = \frac{1}{D^2+a^2}\sin ax$$

$$= \frac{1}{D^2+a^2}\, I.P. \text{ of } (\cos ax + i\sin ax)$$

$$= \frac{1}{D^2+a^2}\, I.P. \text{ of } e^{iax}.$$

$$= I.P. \text{ of } \left[\frac{1}{D^2+a^2}e^{iax}\right] \quad \text{—(1)}$$

now $\frac{1}{D^2+a^2}e^{(ia)x} = e^{iax}\frac{1}{(D+ia)^2+a^2}(1)$

$$= e^{iax}\frac{1}{D^2-a^2+2Dia+a^2}$$

$$= e^{iax}\frac{1}{D^2+2aiD}(1)$$

$$= e^{iax}\frac{1}{2aiD\left(1+\frac{D}{2ai}\right)}(1)$$

$$= \frac{e^{iax}}{2aiD}\left(1+\frac{D}{2ai}\right)^{-1}(1)$$

$$= \frac{e^{iax}}{2aiD}(1) = \frac{e^{iax}}{2ai}(x)$$

$$= -\frac{ix}{2a}(\cos ax + i\sin ax)$$

$$= \frac{x}{2a}(\sin ax - i\cos ax)$$

$\frac{1}{D^2+a^2}e^{iax}$

$\frac{x}{2D}e^{iax}$

$= \frac{x}{2ai}e^{iax}.$

$$= \left[\frac{a\sin ax}{2a} - i\frac{(x\cos ax)}{2a}\right]$$

$\therefore$ I.P of $\left[\frac{1}{D^2+a^2}e^{iax}\right] = \frac{-x\cos ax}{2a}$

$$\therefore \boxed{\frac{1}{D^2+a^2}\sin ax = \frac{-x\cos ax}{2a}}$$

or $\frac{1}{D^2+a^2}\sin ax = \frac{-x\cos ax}{2a}$

$$= \frac{x}{2}\int \sin ax\,dx$$

Sly $\frac{1}{D^2+a^2}\cos ax = \frac{x}{2}\int \cos ax\,dx$

$$= \frac{x}{2}\left[\frac{\sin ax}{a}\right]$$

$$\therefore \boxed{P.I = \frac{1}{D^2+a^2}\sin ax = \frac{-x\cos ax}{2a}}$$

if $f(-a^2)=0$

$$\boxed{P.I = \frac{1}{D^2+a^2}\cos ax = \frac{x\sin ax}{2a}}$$

if $f(-a^2)=0$.

To find P.I when $Q = \sin ax$ (or) $\cos ax$

$\therefore P.I = \frac{1}{D^2+a^2}\sin ax = \frac{-x}{2a}\cos ax$ and $f(-a^2)=0$

$$= \frac{x}{2}\int \sin ax\,dx.$$

$\therefore P.I = \frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax.$

$$= \frac{x}{2}\int \cos ax\,dx.$$

* Solve the following diff. eqns:

$\longrightarrow (D^2+4)y = \sin^2 x.$ $\quad \Big| \sin^2 x = \frac{1-\cos 2x}{2}$

$\longrightarrow (D^2+1)y = \sin 2x \cos x$ | $\sin 3x + \sin x$.

1991 $\rightarrow \left(\frac{d^4y}{dx^4} - m^4 y\right) = \sin x \rightarrow C_1 e^{mx} + C_2 e^{-mx} + (A\cos mx + B\sin mx)$
$+ \frac{1}{1-m^4}\sin x$

1995 $\rightarrow$ find the solution of

$\frac{d^2y}{dx^2} + 4y = 8\cos 2x$. given that $\Rightarrow 2x\sin 2x$.

$y=0$ and $\frac{dy}{dx}=0$ when $x=0$.

1992 $\rightarrow$ solve $(D^2+4)y = \sin 2x$, given that when $x=0$, $y=0$ and $\frac{dy}{dx}=2$.
$\hookrightarrow \frac{9}{8}\sin 2x - \frac{x}{4}\cos 2x$.

1993 $\rightarrow$ solve $(D^4+D^2+1)y = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$

Sol. Given that

$$(D^4+D^2+1)y = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right). \quad \text{—(1)}$$

A.E. of (1) is $D^4+D^2+1=0$

$$\Rightarrow (D^2+1)^2 - D^2 = 0$$

$$\Rightarrow (D^2+1-D)(D^2+1+D) = 0$$

$$\Rightarrow D^2-D+1=0 \;\&\; D^2+D+1=0$$

$$\Rightarrow D = \frac{-1\pm\sqrt{3}i}{2}, \frac{1\pm\sqrt{3}i}{2}$$

$$\therefore y_c = e^{-x/2}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$
$$+ e^{x/2}\left[c_3\cos\left(\frac{\sqrt{3}}{2}x\right) + c_4\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$P.I = \frac{1}{D^4+D^2+1} e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{(D-1/2)^4 + (D-1/2)^2+1}\cos\left(\frac{\sqrt{3}}{2}x\right).$$

$$= e^{-x/2} \frac{1}{D^4 + \frac{1}{16} + D^2 + \frac{D^2}{2} - \frac{D}{2} - 2D^3 + D^2 + \frac{1}{4} - D + 1} \cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2} \frac{1}{D^4 - 2D^3 + \frac{5}{2}D^2 - \frac{3}{2}D + \frac{21}{16}} \cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2} \frac{1}{\left(D^2 + \frac{3}{4}\right)\left(D^2 - 2D + \frac{7}{4}\right)} \cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2} \frac{1}{\left(D^2 + \frac{3}{4}\right)} \left[\frac{1}{-\frac{3}{4} - 2D + \frac{7}{4}} \cos\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$= e^{-x/2} \frac{1}{D^2 + \frac{3}{4}} \cdot \frac{1}{(1 - 2D)} \cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2} \frac{1}{D^2 + \frac{3}{4}} \left[\frac{1 + 2D}{1 - 4D^2} \cos\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$= e^{-x/2} \frac{1}{\left(D^2 + \frac{3}{4}\right)} \left[\frac{1 + 2D}{4} \cos\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$= \frac{e^{-x/2}}{4} \frac{1}{\left[D^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right]} \left[\cos\left(\frac{\sqrt{3}}{2}x\right) - \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$= \frac{e^{-x/2}}{4} \left[\frac{x}{2\left(\sqrt{3}/2\right)} \sin\left(\frac{\sqrt{3}}{2}x\right) + \frac{\sqrt{3}x}{2\left(\sqrt{3}/2\right)} \cos\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$= \frac{e^{-x/2}}{4} \left[\frac{x}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}x + x\cos\frac{\sqrt{3}}{2}x\right]$$

$\therefore y = y_c + y_p$ is the g.s of the given diff. eqn.

Case (vi) To find P.I when $Q = xV$ where $V$ is a fn of $x$

$$P.I = \frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}V$$

Note:— By the repeated use of the above formula $\frac{1}{f(D)}x^m V$ $(m \neq 1)$ can be determined. but it will more tedious.

First operate on $D^2$ if possible otherwise operate on $N^r$ then $D^r$

* Solve the following diff. eqns:

→ $(D^2-1)y = x\sin 3x + \cos x$

→ $(D^2-2D+1)y = x\sin x$

→ $(D^2-2D+1)y = xe^x \sin x$

→ $(D^2+9)y = x\sin x$

→ $(D^2+1)y = e^{-x} + \cos x + x^3 + e^x \sin x$

Note:— If $Q = x^m \sin ax$ (or) $x^m \cos ax$ when the coefficient of $\sin ax$ (or) $\cos ax$ is $x^2$ or $x^3$ or higher power of 'x' then the following method can also be used.

* solve the following diff. eqns.

1. find P.I of $(D^2+1)y = x^2 \sin 2x$

Sol $P.I = \frac{1}{D^2+1} x^2 \sin 2x$

$= \frac{1}{D^2+1} x^2 (\text{I.P. of } e^{2ix})$

$= \text{I.P of } \frac{1}{D^2+1}(e^{2ix} x^2)$ —— (1)

Now $\frac{1}{D^2+1}(e^{2ix} x^2)$

$= e^{2ix} \cdot \frac{1}{(D+2i)^2+1} x^2$

$= e^{2ix} \cdot \left[\frac{1}{D^2+4Di-3}(x^2)\right]$

$= e^{2ix}\left[-\frac{1}{3}\left(1-\left(\frac{D^2}{3}+\frac{4}{3}Di\right)\right)^{-1}(x^2)\right]$

$= -\frac{1}{3} e^{2ix}\left[\left(1+\frac{D^2}{3}+\frac{4D}{3}i-\frac{16}{9}D^2\right)(x^2)\right]$

$= -\frac{1}{3} e^{2ix}\left[x^2+\frac{2}{3}+\frac{4i}{3}(2x)-\frac{16}{9}(2)\right]$

$= -\frac{1}{3} e^{2ix}\left(x^2+\frac{8}{3}xi-\frac{26}{9}\right).$

$= -\frac{1}{3}(\cos 2x + i\sin 2x)\left(x^2+\frac{8}{3}xi-\frac{26}{9}\right).$

$\therefore$ I.P. of $\frac{1}{D^2+1}(e^{2ix} x^2) =$

$-\frac{x^2}{3}(\sin 2x) + \frac{26}{27}\sin 2x$

$-\frac{8x}{9}\cos 2x.$

→ solve $(D^4+2D^2+1)y = x^2\cos x$

→ Solve $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x$.

## case (viii)

If $Q$ is a function of $x$ then (i) $\frac{1}{D-\alpha}Q = e^{\alpha x}\int e^{-\alpha x}Q\,dx$.

(ii) $\frac{1}{D+\alpha}Q = e^{-\alpha x}\int e^{+\alpha x}Q\,dx$.

## problems

→ find P.I of $\frac{1}{D^2-4D+3}(xe^{4x})$

Sol Now $P.I = \frac{1}{D^2-4D+3}(xe^{4x})$

$= \frac{1}{(D-1)(D-3)}(xe^{4x})$

$= \frac{1}{(D-1)}\left[\frac{1}{(D-3)}xe^{4x}\right]$

and so on.

→ $\frac{1}{D^2-5D+6}xe^{4x} = ?$

→ $\frac{1}{6D^2-D-2}xe^{-x} = ?$

→ $\frac{1}{D^2-3D+2}\sin(e^{-x}) = ?$

→ solve $(D^2+a^2)y = \sec ax$

Sol $C.F = c_1 \cos ax + c_2 \sin ax.$

$P.I = \frac{1}{D^2+a^2} \sec ax$

$= \frac{1}{2ia}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right] \sec ax.$ — (1)

Now $\frac{1}{D-ai} \sec ax = e^{iax}\int e^{-iax} \sec ax\, dx$

$= e^{iax}\left[\int (\cos ax - i \sin ax)\frac{1}{\cos ax} dx\right]$

$= e^{iax}\int\left[1 - i \tan ax\right] dx$

$= e^{iax}\left[x + \frac{i}{a}\log(\cos ax)\right]$ — (2)

Replacing $i$ by $-i$ in (2), we get

$\frac{1}{D+ia} \sec ax = e^{-aix}\left[x - \frac{i}{a}\log(\cos ax)\right]$ — (3)

from (1), (2) & (3) we get

$P.I = \frac{1}{2ia}\left[e^{iax}\left(x + \frac{i}{a}\log(\cos ax)\right) - e^{-iax}\left(x - \frac{i}{a}\log(\cos ax)\right)\right]$

$= \frac{x(e^{iax} - e^{-iax})}{2ia} + \frac{1}{a^2}\left(\log(\cos ax)\left(\frac{e^{iax}+e^{-iax}}{2}\right)\right).$

$= \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \log \cos ax.$

$\therefore$ G.S is $y = y_c + y_p$.

→ solve $(D^2+a^2)y = \tan ax$

Ans: $y = (c_1 \cos ax + c_2 \sin ax)$

$-\left(\frac{1}{a^2}\right)\cos ax \log \tan\left(\frac{1}{4}\pi + \frac{1}{2}ax\right).$

→ solve $(D^2+a^2)y = \csc ax$

Ans: $P.I = \frac{-x}{a}\cos ax + \frac{1}{a^2}\sin ax \log(\sin ax)$

→ $(D^2+a^2)y = \cot ax$

Ans: $P.I = \frac{1}{a^2}\sin ax \log\left|\tan\left(\frac{ax}{2}\right)\right|.$

→ find the solution of

$$\frac{d^2 i}{dt^2} + \left(\frac{R}{L}\right)\frac{di}{dt} + \left(\frac{i}{LC}\right) = 0$$

where $R^2C = 4L$ and

$R, C, L$ are constant.

Sol Given that $\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$ —— (1)

$\Rightarrow \left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right)i = 0$ —— (2)

A.E of (2) is

$D^2 + \frac{R}{L}D + \frac{1}{LC} = 0$

$$\therefore D = \frac{-\frac{1}{L} \pm \sqrt{\frac{R^2}{L^2} - 4(1)\left(\frac{1}{LC}\right)}}{2}$$

$$= \frac{1}{2}\left[\frac{-R}{L} \pm \sqrt{\frac{R^2C - 4L}{L^2C}}\right]$$

$$= \frac{1}{2}\left[-\frac{R}{L} \pm \sqrt{\frac{0}{L^2C}}\right]$$

$$(\because R^2C = 4L)$$

$$= \frac{-R}{2L}, -\frac{R}{2L}$$

$$\therefore C.F = (C_1 + C_2 t)\, e^{-\frac{R}{2L}t}$$

$\therefore$ G.S of (1) is

$$i = (C_1 + C_2 t)\, e^{-\frac{R}{2L}t}.$$

→ solve the eqn $\frac{d^2y}{dx^2} = a + bx + cx^2$ given that $\frac{dy}{dx} = 0$, when $x = 0$ and $y = d$ when $x = 0$.

<u>Sol</u> Given that $\frac{d^2y}{dx^2} = a + bx + cx^2$

$$\Rightarrow D^2y = a + bx + cx^2 \quad \text{—— (1)}$$

A.E is $D^2 = 0$

$$\Rightarrow D = 0, 0.$$

$$\therefore \boxed{y_c = C_1 + C_2 x}$$

$$P.I = \frac{1}{D^2}(a + bx + cx^2)$$

$$= \frac{1}{D}\int (a + bx + cx^2)\, dx = \frac{1}{D}\left[ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right]$$

$$= \int \left[ ax + \frac{bx^2}{2} + \frac{cx^3}{3} \right]$$

$$= \frac{ax^2}{2} + \frac{bx^3}{6} + \frac{cx^4}{12}$$

$$y_p = \frac{ax^2}{2} + \frac{bx^3}{6} + \frac{cx^4}{12}.$$

$\therefore$ G.S is $y = y_c + y_p$.

$$\Rightarrow y = (c_1 + c_2 x)e^{0x} + \frac{ax^2}{2} + \frac{bx^3}{6} + \frac{cx^4}{12} \quad \text{—— (2)}$$

$$\Rightarrow \frac{dy}{dx} = c_2 + ax + \frac{bx^2}{2} + \frac{cx^3}{3} \quad \text{—— (3)}$$

putting $y = d$ and $x = 0$

$$d = c_1 + c_2(0) + 0 + 0 + 0$$

$$\Rightarrow \boxed{c_1 = d}$$

$$\therefore (2) \equiv d + \frac{ax^2}{2} + \frac{bx^3}{6} + \frac{cx^4}{12}$$

which is reqd soln of (1).

~~1999~~) If $\frac{d^2x}{dt^2} + \frac{g}{b}(x - a) = 0$ (a, b and g being +ve constants) and $x = a'$ and $\frac{dx}{dt} = 0$ when $t = 0$ then show that $x = a + (a' - a)\cos\left(\sqrt{\frac{g}{b}}\,t\right)$

Sol Given that $\frac{d^2x}{dt^2} + \frac{g}{b}(x - a) = 0$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{g}{b}x = \frac{ga}{b} \quad \text{—— (1)}$$

$$\Rightarrow \left(D^2 + \frac{g}{b}\right)(x) = \frac{ga}{b} \quad \text{where } D = \frac{d}{dt}$$

—— (2)

A.E of (2) is $D^2 + \frac{g}{b} = 0$

$$D^2 = -\frac{g}{b} \Rightarrow D = \pm\sqrt{\frac{g}{b}}\, i$$

$$\therefore \boxed{x_c = c_1 \cos\sqrt{\frac{g}{b}}\,t + c_2 \sin\sqrt{\frac{g}{b}}\,t}$$

$$P.I = \frac{1}{D^2 + \frac{g}{b}}\left(\frac{ga}{b}\right)$$ (continuing next)

Find the solution of the eqn $(D^2-1)y = 1$ which vanishes $(y=0)$ when $x=0$ and tends to a finite limit as $x \to -\infty$ and $D$ stands for $\frac{d}{dx}$.

<u>Sol</u>. $(D^2-1)y = 1$ —— (1)

A.E is $D^2 - 1 = 0$

$D = \pm 1$

$$\therefore \boxed{y_c = c_1 e^x + c_2 e^{-x}}$$

$$P.I = \frac{1}{D^2-1}(1) = \frac{1}{D^2-1}e^{0x}$$

$$= \frac{1}{-1} = -1$$

$$\therefore \boxed{y_p = -1}$$

G.S is $y = y_c + y_p$

$$\boxed{y = c_1 e^{x} + c_2 e^{-x} - 1} \quad \text{—(2)}$$

putting $y=0$ and $x=0$ in (1), we get

$$0 = c_1 + c_2 - 1$$

$$\Rightarrow \boxed{c_1 + c_2 = 1} \quad \text{—(3)}$$

multiplying both sides of (2). by $e^{x}$,

we get $y e^{x} = c_1 (e^{x})^2 + c_2 - e^{x}$

Taking limit on both sides of (2)

as $x \to -\infty$ we get

$$\lim_{x \to -\infty} y e^{x} = \lim_{x \to -\infty} c_1 (e^{x})^2 + \lim_{x \to \infty} c_2 - \lim_{x \to \infty} e^{x}$$

$$\therefore y \times 0 = c_1 (0)^2 + c_2 - 0$$

$$\left(\because \lim_{x \to \infty} e^{x} = 0\right)$$

$$\Rightarrow \boxed{c_2 = 0}$$

$$\therefore (3) \equiv \boxed{c_1 = 1}$$

$$(2) \equiv \boxed{y = e^{x} - 1}$$

which is the reqd soln.

2001
1997 → solve $(D^2+1)^2 y = 24x \cos x$ given the initial conditions $x=0, y=0, Dy=0, D^2y=0$

$D^3 y = 12$

Ans: $y = 3x^2 \sin x - x^3 \cos x$.

1993, solve $\frac{d^2y}{dx^2} + w_0^2 y = a \cos wt$ and discuss the nature of solution as $w \to w_0$

---

1995, Determine all real valued solutions of the eqn $y''' - i y'' + y' - i y = 0$; $y' = \frac{dy}{dx}$.

Sol Given that $y''' - i y'' + y' - i y = 0$

$$\Rightarrow \frac{d^3y}{dx^3} - i \frac{d^2y}{dx^2} + \frac{dy}{dx} - i y = 0$$

$$\Rightarrow [D^3 - i D^2 + D - i] y = 0 \qquad D = \frac{d}{dx}.$$

——— (1)

A.E is $D^3 - i D^2 + D - i = 0$

$$\Rightarrow D^2 (D - i) + (D - i) = 0$$

$$\Rightarrow D = i \;;\; D = \pm i$$

(continuing next)

# CAUCHY-EULER EQUTIONS

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X$$

where $a_1, a_2, \ldots, a_n$ are constants and $X$ is a function of $x$, is called the cauchy-Euler homogeneous linear equation of the $n^{th}$ order.

Method of solution:-

Reduce the linear equation

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X$$

into linear equation with constant coeffecients.

The given equation is

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X \quad \text{--- (1)}$$

put $x = e^z \Rightarrow z = \log x \quad (x > 0)$

$$\Rightarrow \frac{dz}{dx} = \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x}$$

$$\therefore x\frac{dy}{dx} = \frac{dy}{dz} \Rightarrow \boxed{x\frac{dy}{dx} = D_1 y}$$

where $D_1 = \frac{d}{dz}$

$$\text{Now } \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$$

$$= \frac{d}{dx}\left[\frac{1}{x}\frac{dy}{dz}\right]$$

$$= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) + \frac{dy}{dz}\left(\frac{d}{dx}\left(\frac{1}{x}\right)\right)$$

$$=\frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dx}\right)+\frac{dy}{dz}\left(-\frac{1}{x^2}\right)$$

$$=\frac{1}{x}\frac{d}{dz}\left(\frac{1}{x}\frac{dy}{dz}\right)+\frac{dy}{dz}\left(-\frac{1}{x^2}\right)$$

$$=\frac{1}{x^2}\frac{d^2y}{dz^2}-\frac{1}{x^2}\frac{dy}{dz}$$

$$=\frac{1}{x^2}(D_1^2-D_1)y \quad \text{where } D_1=d/dz$$

$$=\frac{1}{x^2}D_1(D_1-1)y..$$

$$\therefore \boxed{x^2\frac{d^2y}{dx^2}=D_1(D_1-1)y}$$

sly $x^3\dfrac{d^3y}{dx^3}=D_1(D_1-1)(D_1-2)y$

$\vdots$

$$x^n\frac{d^ny}{dx^n}=D_1(D_1-1)(D_1-2)\text{------}(D_1-(n-1))y.$$

$\therefore$ From ①, we have

$$[D_1(D_1-1)(D_1-2)\text{-----}(D_1-(n-1))]+a_1D_1(D_1-1)(D_1-2)\text{----}$$
$$\text{--}(D_1-(n-2))+\text{----}+a_{n-1}D_1+a_n]y=e^z$$

clearly which is a linear equation with constant Coefficients and is therefore solvable for $y$ interms of $z$.

→ If $y=F(z)$ is its solution then putting $z=\log x$, then the required solution is

$$y=F(\log x)$$

Problems

→ Solve the following:

(1) $(x^2D^2-xD+2)y=x\log x$

Sol: Given that $(x^2D^2-xD+2)y=x\log x \longrightarrow$ ①

Putting $x=e^z \Rightarrow z=\log x$

Let $D_1=\dfrac{d}{dz}$ then

from ①, we have

$$(D_1(D_1-1)-D_1+2)y=e^z z$$

$$\Rightarrow (D_1^2-2D_1+2)y=ze^z \longrightarrow ②$$

A.E of ② is

$D_1^2 - 2D_1 + 2 = 0$

$D_1 = \frac{2 \pm \sqrt{4 - 4(1)2}}{2(1)}$

$= \frac{2 \pm \sqrt{-4}}{2}$

$= \frac{2 \pm 2i}{2} = 1 \pm i$

$\therefore \boxed{y_c = e^z (c_1 \cos z + c_2 \sin z)}$

P.I of ② is $\frac{1}{D_1^2 - 2D_1 + 2} (e^z z)$

$= e^z \left[ \frac{1}{(D_1+1)^2 - 2(D_1+1) + 2} (z) \right]$

$= e^z \left[ \frac{1}{D_1^2 + 1 + 2D_1 - 2D_1 - 2 + 2} z \right]$

$= e^z \left[ \frac{1}{D_1^2 + 1} z \right]$

$= e^z (1 + D_1^2)^{-1} z$

$= e^z (1 - D_1^2 + \cdots) z$

$= e^z (z - 0)$

$= z e^z$

$\therefore \boxed{y_p = z e^z}$

G.S of ② is $y = y_c + y_p$

$y = e^z (c_1 \cos z + c_2 \sin z) + z e^z$

$\Rightarrow y = x (c_1 \cos(\log x) + c_2 \sin(\log x)) + x \log x$

which is the required general solution of ①

H.W
(2) $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$

Note:- $x^n\frac{d^ny}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}x\frac{dy}{dx} + a_n y = x^m$

Let $x = e^z$ then convert only LHS interms of z by using $D_1 = \frac{d}{dz}$ but doesnot change RHS $x = x^m$ interms of z

$\therefore$ The given equation reduces to $f(D_1)y = x^m$

For this special case, we use the following formula directly. $\frac{1}{f(D_1)}x^m = \frac{1}{f(m)}x^m$, provided $f(m) \neq 0$.

Problem :

$\rightarrow$ solve $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x$ —— ①

Sol: Let $x = e^z \Rightarrow z = \log x$ and

let $D_1 = \frac{d}{dz}$

then $[D_1(D_1-1) - 4D_1 + 6]y = x$

$\Rightarrow [D_1^2 - D_1 - 4D_1 + 6]y = x$

$\Rightarrow [D_1^2 - 5D_1 + 6]y = x$ —— ②

A.E of ② is $D_1^2 - 5D_1 + 6 = 0$

$\Rightarrow D_1 = 2, 3$

$\therefore \boxed{y_c = c_1 e^{2z} + c_2 e^{3z}}$

$y_p = \frac{1}{D_1^2 - 5D_1 + 6}(x^1) = \frac{1}{1-5+6}(x)$

$= \frac{1}{2}(x)$

$\therefore$ The g.s ① is $y = y_c + y_p$

$y = c_1 e^{2z} + c_2 e^{3z} + \frac{1}{2}(x)$

$\Rightarrow \boxed{y = c_1 x^2 + c_2 x^3 + x/2}$

which is the required g.s of ①

Note:- An alternative method for getting P.I of Cauchy-Euler equation without changing the R.H.S interms of Z.

Let $F(D_1)y = f(x)$ where $D_1 = \frac{d}{dz}$.

then $\frac{1}{D_1-\alpha} f(x) = x^{\alpha}\int x^{-\alpha-1} f(x)dx$.

and $\frac{1}{D_1+\alpha} f(x) = x^{-\alpha}\int x^{\alpha-1} f(x)dx$.

To evaluate P.I, we first factorize $F(D_1)$ into linear factors and then one of the following methods can be used.

Method(I): If the operator $\frac{1}{F(D_1)}$ into partial fractions then

$$P.I = \frac{1}{F(D_1)} f(x) = \left[\frac{A_1}{(D_1-\alpha_1)} + \frac{A_2}{(D_1-\alpha_2)} + \cdots + \frac{A_n}{D_1-\alpha_n}\right] f(x)$$

Method(II): $P.I = \frac{1}{(D_1-\alpha_1)(D_1-\alpha_2)\cdots(D_1-\alpha_n)} f(x)$

where the operations indicated by factors are to be taken in succession, begining with the first on the right.

Problems:-

→ solve $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x + \sin x$

Sol:- Given that $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x + \sin x$ ——— ①

Let $x = e^z \Rightarrow z = \log x$

then ① $= [D_1(D_1-1) + 4D_1 + 2]\,y = x + \sin x$

$\Rightarrow [D_1^2 + 3D_1 + 2]\,y = x + \sin x$ ——— ②

Now A.E of ② is $m_1^2 + 3m_1 + 2 = 0$

$\Rightarrow m = -2, -1$

$\therefore \boxed{y_c = c_1 e^{-2z} + c_2 e^{-z}}$

$P.I = \frac{1}{D_1^2+3D_1+2}(x + \sin x) = \frac{1}{(D_1+2)(D_1+1)}(x + \sin x)$

$= \frac{1}{(D_1+2)(D_1+1)} x + \frac{1}{(D_1+2)(D_1+1)} \sin x$ ——— ③

Now $\frac{1}{(D_1+2)(D_1+1)}\sin x = \frac{1}{D_1+2}\left[\frac{1}{D_1+1}\sin x\right]$

$$= \frac{1}{D_1+2}\left[x^{-1}\int x^{1-1}\sin x\,dx\right]$$

$$= \frac{1}{D+2}\left[\frac{-1}{x}\cos x\right]$$

$$= x^{-2}\int x^{2-1}\left(\frac{-1}{x}\cos x\right)dx$$

$$= -x^{-2}\int \cos x\,dx$$

$$= -\frac{1}{x^2}\sin x$$

$\therefore$ G.S of ② is $y = y_c + y_c$

$$y = (c_1 e^{-2z} + c_2 e^{-z}) - \frac{1}{x^2}\sin x + \frac{x}{6}$$

$$\Rightarrow \boxed{y = c_1 x^{-2} + c_2 x^{-1} + \frac{x}{6} - \frac{1}{x^2}\sin x.}$$

which is reqd g.s.

* solve the following diff. eqns :-

→ $(x^2D^2 + xD - 1)y = x e^{2x}$ Ans: $y = c_1 x + c_2 x^{-1} + \frac{1}{8}(e^{2x})(2 - x^{-1})$

→ $(x^2D^2 + xD - 1)y = x^2 e^x$ Ans: $y = c_1 x + c_2 x^{-1} + e^x(1 - x)$

→ $x^3\frac{d^3y}{dx^3} + 2x^2\left(\frac{d^2y}{dx^2}\right) + 2y = 10\left(x + \frac{1}{x}\right)$

96 → $(x^3D^3 + 3x^2D^2 + xD + 1)y = x\log x$

1990 → $x^2\left(\frac{d^2y}{dx^2}\right) + 2x\left(\frac{dy}{dx}\right) + y = 0$

→ $x^2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 2y = e^x$

* LEGENDRE'S LINEAR EQUATIONS:

An equation of the form

$$(a+bx)^n\frac{d^ny}{dx^n} + A_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + A_2(a+bx)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots$$

$$+ A_{n-1}(ax+b)\frac{dy}{dx} + A_n y = Q(x).$$

where $A_1, A_2, \ldots A_n$ are constants and $Q(x)$ is a function of 'x' is called Legendre's linear equation of the $n^{th}$ order.

* Method of solution:-

Reduce the linear equation —

$$(a+bx)^n \frac{d^n y}{dx^n} + A_1 (a+bx)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + A_2(a+bx)^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \cdots$$

$$+ A_{n-1}(ax+b)\frac{dy}{dx} + A_n y = Q(x) \quad \text{——— (1)}$$

where $A_1, A_2, \ldots A_{n-1}, A_n$ are constants. into linear equation with constant coefficients.

Putting $a+bx = e^z \Rightarrow z = \log(a+bx)$

$$\Rightarrow \frac{dz}{dx} = b\left(\frac{1}{a+bx}\right).$$

Now $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$

$$= \frac{dy}{dz}\left[b \frac{1}{a+bx}\right]$$

$$(a+bx)\frac{dy}{dx} = b\frac{dy}{dz}$$

$$\Rightarrow \boxed{(a+bx)\frac{dy}{dx} = bD_1 y} \quad \text{Where } D_1 = \frac{d}{dz}$$

Now $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$

$$= \frac{d}{dx}\left[\frac{dy}{dz} \frac{b}{a+bx}\right]$$

$$= \frac{b}{a+bx}\frac{d}{dx}\left(\frac{dy}{dz}\right) + \frac{dy}{dz}\frac{d}{dx}\left(\frac{b}{a+bx}\right).$$

$$= \frac{b}{a+bx}\frac{d}{dz}\left(\frac{dy}{dx}\right) + \frac{dy}{dz}\left(\frac{-b^2}{(a+bx)^2}\right)$$

$$= \frac{b}{a+bx}\frac{d}{dz}\left[\frac{dy}{dz}\cdot\frac{b}{a+bx}\right] - \frac{b^2}{(a+bx)^2}\frac{dy}{dz}$$

$$= \frac{b^2}{(a+bx)^2}\frac{d^2y}{dz^2} - \frac{b^2}{(a+bx)^2}\frac{dy}{dz}$$

$$= \frac{b^2}{(a+bx)^2}(D_1^2 - D_1)y$$

$$= \frac{b^2}{(a+bx)^2} D_1(D_1-1)y.$$

$$\therefore \boxed{(a+bx)^2 \frac{d^2y}{dx^2} = b^2 D_1(D_1-1)y}$$

Sly $(a+bx)^3 \frac{d^3y}{dx^3} = b^3 D_1(D_1-1)(D_1-2)y$

$$(a+bx)^n \frac{d^ny}{dx^n} = b^n[D_1(D_1-1)(D_1-2)\cdots(D_1-(n-1))]y$$

$\therefore$ from (1), we have,

$b^n(D_1(D_1-1)(D_1-2)\cdots\cdots(D_1-(n-1)))+$
$b^{n-1}A_1(D_1(D_1-1)(D_1-2)\cdots(D_1-(n-2)))+\cdots\cdots+A_{n-1}bD_1$
$+A_n]y = Q\left(\frac{e^z-a}{b}\right)$

clearly which is linear equation with constant coefficients and hence is solvable for $y$ interms of $z$.

Working rule:
putting $a+bx = e^z \Rightarrow z = \log(a+bx)$

Let $D_1 = \frac{d}{dz}$ then

$$(a+bx)\frac{dy}{dx} = bD_1y$$

$$(a+bx)^2\frac{d^2y}{dx^2} = b^2D_1(D_1-1)y \text{ etc.}$$

Problems:

→ Solve the following:

(1) $[(3x+2)^2D^2+3(3x+2)D-36]y = 3x^2+4x+1$

Sol: Given that

$$\left[(3x+2)^2D^2+3(3x+2)D-36\right]y = 3x^2+4x+1 \quad \text{——— (1)}$$

putting $3x+2 = e^z \Rightarrow z = \log(3x+2)$

Let $D_1 = \frac{d}{dz}$ then

$$(3x+2)\frac{dy}{dx} = 3D_1y$$

$$(3x+2)^2\frac{d^2y}{dx^2} = 3^2D_1(D_1-1)y$$

From (1), we have

$$\left[3^2D_1(D_1-1)+3(3)D_1-36\right]y = \frac{e^{2z}-1}{3}$$

$$\Rightarrow 9\left[D_1^2-\cancel{D_1}+\cancel{D_1}-4\right]y = \frac{e^{2z}-1}{3}$$

$$\Rightarrow \left[D_1^2-4\right]y = \frac{e^{2z}-1}{27} \quad \text{——— (2)}$$

A.E of (2) is $m^2-4=0$

$m^2=4 \Rightarrow m=\pm 2$

$$\therefore \boxed{y_c = c_1e^{-2z}+c_2e^{+2z}}$$

P.I of (2) is $\frac{1}{D_1^2-4}\left(\frac{e^{z}-1}{2z}\right) = \frac{1}{27}\left[\frac{1}{D_1^2-4}(e^{2z}-1)\right]$

$$= \frac{1}{27}\left[\frac{1}{D_1^2-4}e^{2z} - \frac{1}{D_1^2-4}(1)\right]$$

$$= \frac{1}{27}\left[e^{2z}\frac{1}{(D_1+2)^2-4}(1) - \frac{1}{-4}\right]$$

$$= \frac{1}{27}\left[e^{2z}\frac{1}{D_1^2+2D_1}(1) + \frac{1}{4}\right]$$

$$= \frac{1}{27}\left[e^{2z}\frac{z}{2D_1+2}(1) + \frac{1}{4}\right]$$

$$= \frac{1}{27}\left[e^{2z}\frac{z}{2}(1) + \frac{1}{4}\right]$$

$$= \frac{1}{27}\left[\frac{ze^{2z}}{2} + \frac{1}{4}\right]$$

$$= \frac{1}{54}\left[ze^{2z} + \frac{1}{2}\right]$$

$\therefore$ G.S of (2) is $y = y_c + y_p$

$$y = (c_1 e^{-2z} + c_2 e^{2z}) + \frac{1}{54}(z e^{2z} + \tfrac{1}{2})$$

$$\Rightarrow y = \left[c_1 \frac{1}{(3x+2)^2} + c_2 (3x+2)^2\right] + \frac{1}{54}\left[\log(3x+2)\cdot(3x+2)^2 + \frac{1}{2}\right]$$

which is the reqd g.s of ①

$\xrightarrow{\text{H.W}}$ $(x+1)^2 y_2 - 3(x+1)y_1 + 4y = x^2$

$\rightarrow$ $[(5+2x)^2 D^2 - 6(5+2x)D + 8]y = 0$

1997 H.W $(1+2x)^2 \frac{d^2y}{dx^2} - 6(1+2x)\frac{dy}{dx} + 16y = 8(1+2x)^2$

$y(0) = 0,\ y'(0) = 2$

$\xrightarrow{\text{H.W}}$ $[(x+1)^4 D^3 + 2(x+1)^3 D^2 - (D+1)^2 D + (x+1)]y = \frac{1}{x+1}$

* **Linear Differential Equations of the second order with variable coeffecients**

An equation of the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x).$$

where $P(x)$, $Q(x)$ and $R(x)$ are functions of '$x$' is called linear diff. equation of the second order with variable coefficients.

$\rightarrow$ The linear diff. equation of the second order with variable coefficients can be solved by the following methods.

(1) Change of the dependent variable, when a part of the complementary function (C.F) is known.

(2) Change of the dependent variable and removal of the first derivative (or) reduction to normal form (or) canonical form

(3) Change of the independent variable.

(4) Method of Variation of parameters.

1. To Solve $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$ by Changing of the dependent variable when a part of the C.F is Known:-

Given equation is-

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x) \quad \text{--- (1)}$$

and its linear homogenous equation is

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0 \quad \text{--- (2)}$$

Let $y=u(x)$ be a Known solution of the C.F of (1)

Then. $y=u(x)$ is a solution of (2),

$$\frac{d^2u}{dx^2}+P(x)\frac{du}{dx}+Q(x)u(x)=0 \quad \text{--- (3)}$$

Let $y=uv$ be the g.s of (1) where $u=u(x)$, $v=v(x)$

then $$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx} \quad \text{--- (4)}$$

$$\frac{d^2y}{dx^2}=u\frac{d^2v}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+v\frac{d^2u}{dx^2} \quad \text{--- (5)}$$

$$\therefore (1) \equiv u\frac{d^2v}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+v\frac{d^2u}{dx^2}+P(x)\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)+Q(x)uv=R(x).$$

$$\Rightarrow v\left[\frac{d^2u}{dx^2}+P(x)\frac{du}{dx}+Q(x)u\right]+u\left[\frac{d^2v}{dx^2}+P(x)\frac{dv}{dx}\right]+2\frac{du}{dx}\frac{dv}{dx}=R(x)$$

$$\Rightarrow v\cdot[0]+u\left(\frac{d^2v}{dx^2}+P(x)\frac{dv}{dx}\right)+2\frac{du}{dx}\frac{dv}{dx}=R(x)\ (\text{by } (3))$$

$$\Rightarrow u\frac{d^2v}{dx^2}+uP(x)\frac{dv}{dx}+2\frac{du}{dx}\frac{dv}{dx}=R(x)$$

$$\Rightarrow u\frac{d^2v}{dx^2}+\left(uP(x)+2\frac{du}{dx}\right)\frac{dv}{dx}=R(x).$$

$$\Rightarrow \frac{d^2v}{dx^2}+\left(P(x)+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R(x)}{u} \longrightarrow (6)$$

Let $\frac{dv}{dx} = V$ then $\frac{d^2v}{dx^2} = \frac{dV}{dx}$

$$\therefore ⑥ \equiv \frac{dV}{dx} + \left(P(x) + \frac{2}{u}\frac{du}{dx}\right)V = \frac{R(x)}{u} \quad \text{--- } ⑦$$

Clearly this is linear equation in V of first order

$$\therefore I.F = e^{\int \left(P(x) + \frac{2}{u}\frac{du}{dx}\right)dx}$$

$$= e^{\int P(x)dx + \int \frac{2}{u}du}$$

$$= e^{\int P(x)dx + 2\log u}$$

$$\boxed{I.F = u^2 e^{\int P(x)dx}}$$

$\therefore$ The g.s of ⑦ is

$$V.u^2 e^{\int P(x)dx} = \int \left[\frac{R}{u} u^2 e^{\int P(x)dx}\right] dx + c$$

$$\Rightarrow Vu^2 e^{\int P(x)dx} = \int \left[Ru e^{\int P(x)dx}\right] dx + c$$

$$\Rightarrow \frac{dv}{dx} u^2 e^{\int P(x)dx} = \int \left[Ru e^{\int P(x)dx}\right] dx + c \quad \left(\because V = \frac{dv}{dx}\right)$$

$$\Rightarrow \frac{dv}{dx} = \frac{e^{-\int P(x)dx}}{u^2}\left[\int \left(Ru e^{\int P(x)dx}\right)dx\right] + \frac{ce^{-\int P(x)dx}}{u^2}$$

$$\Rightarrow v = \int \left[\left[\frac{1}{u^2} e^{-\int P(x)dx} \int \left(Ru e^{\int Pdx}\right)dx + \frac{ce^{-\int P(x)dx}}{u^2}\right]dx + c_2\right.$$

$\therefore$ From ⓘ, we have

$$y = c_2 u + u \int \left[\frac{1}{u^2} e^{-\int P(x)dx} \int \left(Ru e^{\int Pdx}\right)dx + \left[\frac{ce^{-\int P(x)dx}}{u^2}\right]dx\right.$$

Since this solution includes the known solution $y = u(x)$ and it contains two arbitrary constants, It is the g.s of ①

*Methods for finding one integral (solution) in C.F by inspection i.e. Solution of $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$

**Method (1):**

If $y=e^{ax}$ is a solution of

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0 \quad \text{——— (i)}$$

then $\frac{dy}{dx}=ae^{ax}$ ; $\frac{d^2y}{dx^2}=a^2e^{ax}$

$$\therefore (i) \equiv a^2e^{ax}+P(x).ae^{ax}+Q(x)e^{ax}=0$$

$$\Rightarrow a^2+aP(x)+Q(x)=0 \quad (\because e^{ax}\neq 0) \text{ ——— (ii)}$$

$\therefore$ If $y=e^{ax}$ is a solution of (i) then $a^2+aP(x)+Q(x)=0$

**putting $a=1$:**

$\therefore y=e^{x}$ is a solution of (i) then $1+P(x)+Q(x)=0$

**Putting $a=-1$:**

$\therefore y=e^{-x}$ is a solution of (i)

then $1-P+Q=0$

$\therefore$ if $1+P+Q=0$ then $y=e^{x}$ is a solution of (i)

if $1-P+Q=0$ then $y=e^{-x}$ is a solution of (i).

i.e a part of the C.F of $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)=R(x)$

**Method (2):**

If $y=x^m$ is a solution of $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$

then $\frac{dy}{dx}=mx^{m-1}$ ; $\frac{d^2y}{dx^2}=m(m-1)x^{m-2}$

$$\therefore (1) \equiv m(m-1)x^{m-2}+P(x)mx^{m-1}+Q(x)x^m=0$$

$$\Rightarrow m(m-1)+P(x)mx+Q(x)x^2=0$$

$\therefore$ If $y=x^m$ is a solution of (i) then

$$m(m-1)+P(x)mx+Q(x)x^2=0$$

**putting** $m=1$ in the above

$\therefore y=x$ is a solution of (i) then

$$0+xP(x)+x^2Q(x)=0$$

$\boxed{P + xQ = 0}$ $(\because x \neq 0)$

Putting $m = 2$

$\therefore y = x^2$ is a solution of (i) then

$2(2-1) + 2P.x + Q.x^2 = 0$

$\Rightarrow \boxed{2 + 2P.x + Q.x^2 = 0}$

Conversely suppose that $P + Qx = 0$ then $y = x$ is solution of (i)

and $2 + 2Px + Q.x^2 = 0$ then $y = x^2$ is solution of (i)

Working rule:

step(1): write the given equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x) = R(x)$$

step(2): Find one solution $u(x)$ of C.F by using the following formula

| Condition satisfied | one solⁿ of C.F is |
|---|---|
| (1) $a^2 + aP + Q = 0$ | $u = e^{ax}$ |
| (i) $1 + P + Q = 0$ | $u = e^{x}$ |
| (ii) $1 - P + Q = 0$ | $u = e^{-x}$ |
| (2) $m(m-1) + Pmx + Qx^2 = 0$ | $u = x^m$ |
| (i) $P + Qx = 0$ | $u = x$ |
| (ii) $2 + 2Px + Qx^2$ | $u = x^2$ |

Step(3): Assume the g.s of given equation is $y = uv$ where $u$ is obtained by step(2) and $v$ is obtained by

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}.$$

Problems: Solve the following:

$\rightarrow xy'' - (2x-1)y' + (x-1)y = 0$ —— ①

$\Rightarrow \frac{d^2y}{dx^2} - (2 - 1/x)y' + \left(1 - \frac{1}{x}\right)y = 0$ —— ②

Comparing ① with

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

$P = -(2 - \frac{1}{x})$; $Q = (1 - 1/x)$ and $R = 0$

Now here

$$1 + P + Q = 1 - 2 + \frac{1}{x} + 1 - \frac{1}{x} = 0$$

$\therefore y = u = e^{x}$ is a part of C.F of ①

Let $y = uv$ — ② be the g.s of ①

then $v$ is given by

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u} \quad \text{— ③}$$

Since $u = e^{x} \Rightarrow \frac{du}{dx} = e^{x}$;

$$P + \frac{2}{u}\frac{du}{dx} = -2 + \frac{1}{x} + 2e^{-x}e^{x}$$

$$= -2 + \frac{1}{x} + 2$$

$$= 1/x$$

$\therefore$ From ③, we have

$$\frac{d^2v}{dx^2} + \left(\frac{1}{x}\right)\frac{dv}{dx} = 0 \quad \text{— ④}$$

Take $\frac{dv}{dx} = V \Rightarrow \frac{d^2v}{dx^2} = \frac{dV}{dx}$

$\therefore$ ④ $\equiv \frac{dV}{dx} + \frac{1}{x}V = 0$

$$\Rightarrow \frac{dV}{V} = -\frac{1}{x}dx$$

$$\Rightarrow \log(Vx) = \log c_1$$

$$\Rightarrow Vx = c_1$$

$$\Rightarrow \frac{dv}{dx}x = c_1 \quad \left(\because V = \frac{dv}{dx}\right)$$

$$\Rightarrow dv = \frac{c_1}{x}dx$$

$$\Rightarrow \boxed{v = c_1 \log x + c_2}$$

$\therefore$ ② $\equiv y = e^{x}(c_1 \log x + c_2)$ is the reqd g.s of ①.

→ $y'' + y = \sec x$ given that $\cos x$ is a part of C.F

Sol: Given that $\frac{d^2y}{dx^2} + y = \sec x$ —— ①

and $y = u = \cos x$ is a part of C.F of ①

Comparing (1) with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

$P = 0;\ Q = 1;\ R = \sec x.$

Let $y = uv$ be the g.s of ①

then $v$ is obtained by

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u} \text{ —— ②}$$

since $u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$

$$\therefore P + \frac{2}{u}\frac{du}{dx} = 0 + \frac{2}{\cos x}(-\sin x)$$

$$= -2\tan x$$

Now from ②, we have

$$\frac{d^2v}{dx^2} - 2\tan x\frac{dv}{dx} = \frac{\sec x}{\cos x}.$$

$$\Rightarrow \frac{d^2v}{dx^2} - 2\tan x\frac{dv}{dx} = \sec^2 x \text{ —— ③}$$

Let $\frac{dv}{dx} = V \Rightarrow \frac{d^2v}{dx^2} = \frac{dV}{dx}$

$$\therefore ③ \equiv \frac{dV}{dx} - 2\tan x\, V = \sec^2 x \text{ —— ④}$$

$$\text{I.F} = e^{-\int 2\tan x\, dx}$$

$$= e^{+2\log(\cos x)}$$

$$= \cos^2 x$$

$\therefore$ G.s of ④ is $V\cos^2 x = \int \sec^2 x \cos^2 x\, dx + c_1$

$$= x + c_1$$

$$\therefore V\cos^2 x = x + c_1$$

$$\Rightarrow \frac{dv}{dx}\cos^2 x = x + c_1 \quad \left(\because V = \frac{dv}{dx}\right)$$

$$\Rightarrow dv = (x + c_1)\sec^2 x\, dx.$$

$\Rightarrow v = \int (x+c_1)\sec^2 x\,dx + c_2$

$= (x+c_1)\tan x + \log(\cos x) + c_2$

$= \dfrac{(x+c_1)\sin x + \cos x \log(\cos x) + c_2 \cos x}{\cos x}$

$\therefore$ G.S of (1) is $y = uv$

$\Rightarrow y = (x+c_1)\sin x + \cos x \log(\cos x) + c_2 \cos x$

H.W $xy'' - (2x+1)y' + (x+1)y = x^3 e^x$

H.W $(1-x^2)y_2 + xy_1 - y = x(1-x^2)^{3/2}$ $\quad -\frac{1}{9}(1-x^2)^{3/2} - c_1\left[\frac{\sqrt{1-x^2}}{x} + \sin^{-1}(x)\right] + c_2$

H.W $xy_2 - y_1 - 4x^3 y = -4x^5$, given that $y = e^{x^2}$ is a solution if that left hand side is equated to zero.

Ans: $\boxed{y = c_1 e^{-x^2} + c_2 e^{x^2} + x^2}$ $\quad \boxed{-x^2 + \left(\frac{c_1}{4}\right)e^{-x^2} + c_2 e^{x^2}}$

(2) To solve :- $\dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx} + Q(x)y = R(x)$ by changing the dependent variable and removal of the first derivative.

[2000] (or) Reduce the diff. equation $y'' + Py' + Qy = R$ where $P, Q, R$ are functions of $x$, to the form $\dfrac{d^2v}{dx^2} + Iv = S$ which is known as the normal form of the given equation.

Sol:- Given equation is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Let $y = uv$ —— (2) be the g.s of (1) where $u, v$, are fns of $x$.

Now $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ —— (3)

and $\dfrac{d^2y}{dx^2} = u\dfrac{d^2v}{dx^2} + 2\dfrac{du}{dx}\dfrac{dv}{dx} + v\dfrac{d^2u}{dx^2}$ —— (4)

(1) $\Rightarrow u\dfrac{d^2v}{dx^2} + 2\dfrac{du}{dx}\dfrac{dv}{dx} + v\dfrac{du}{dx} + P\left(u\dfrac{dv}{dx} + v\dfrac{du}{dx}\right) + Quv = R$

$\Rightarrow u\dfrac{d^2v}{dx^2} + \left(Pu + 2\dfrac{du}{dx}\right)\dfrac{dv}{dx} + \left(\dfrac{d^2u}{dx^2} + P\dfrac{du}{dx} + Qu\right)v = R$ —— (5)

To remove the first derivative $\dfrac{dv}{dx}$ in (5)

choose $u$ s.t $Pu + 2\dfrac{du}{dx} = 0$ —— (6)

$\Rightarrow \dfrac{du}{dx} = -P/2\, u$

$\Rightarrow \frac{du}{u} = -\frac{1}{2} P(x)\,dx$

$\Rightarrow \log u = -\frac{1}{2}\int P(x)\,dx$

$\Rightarrow \boxed{u = e^{-\frac{1}{2}\int P\,dx}}$

(6) $\equiv \boxed{\frac{du}{dx} = -\frac{1}{2}Pu}$ — (7)

$$\frac{d^2u}{dx^2} = -\frac{1}{2}P\frac{du}{dx} - \frac{1}{2}u\frac{dP}{dx}$$

$$\boxed{\frac{d^2u}{dx^2} = -\frac{1}{2}P\left(-\frac{1}{2}Pu\right) - \frac{1}{2}u\frac{dP}{dx}} \quad (8)$$

(5) $\equiv u\frac{d^2v}{dx^2} + \left(Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}\right)uv = R$

$\Rightarrow \frac{d^2v}{dx^2} + \left(Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}\right)v = \frac{R}{u}$

$\Rightarrow \boxed{\frac{d^2v}{dx^2} + Iv = S}$ — (9)

where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$, $S = \frac{R}{u}$ which is the normal form of (1)

The g.s of (1) is $y = uv$ where $u = e^{-\frac{1}{2}\int P\,dx}$ & $v$ is given by (9)

Working rule:

Step (1): Write the given equation in the standard form

$y'' + Py' + Qy = R$

Step (2): To remove the first derivative we choose $u = e^{-\frac{1}{2}\int P\,dx}$.

Step (3): Assume the g.s of the given eqn is $y = uv$ then $u$ is given by step (2) and $v$ is given by the normal form. $\frac{d^2v}{dx^2} + Iv = S$

where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$ and $S = \frac{R}{u}$

Problems.

→ Solve the following:

2000 (1) $y'' - 4xy' + (4x^2-1)y = -3e^{x^2}\sin 2x.$

Sol: Comparing (1) with $y'' + P(x)y' + Q(x)y = R(x)$

$$P = -4x \; ; \; Q = 4x^2-1 \; ; \; R = -3e^{x^2}\sin 2x$$

To remove the first derivative

we choose $u = e^{-\frac{1}{2}\int P\,dx}$

$$= e^{-\frac{1}{2}\int(-4x)\,dx}$$

$$= e^{x^2} \quad \text{---} (2)$$

Let $y = uv$ — (3) be the g.s of (1)

then $v$ is given by the normal form $\frac{d^2v}{dx^2} + Iv = S$ — (4)

where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$; $S = \frac{R}{u}$

Now $I = 4x^2 - 1 - \frac{1}{4}(16x^2) - \frac{1}{2}(-4)$
$= 1$

and $S = -3\sin 2x$

$\therefore$ (4) $\equiv \frac{d^2v}{dx^2} + v = -3\sin 2x$

$\Rightarrow (D^2+1)v = -3\sin 2x$ — (5)

A.E is $D^2+1 = 0$

$\Rightarrow D = \pm i$

$\therefore$ C.F $= c_1\cos 2x + c_2\sin 2x$

$$P.I = \frac{1}{D^2+1}(-3\sin 2x)$$

$$= \frac{-3}{-3}\sin 2x$$

$$= \sin 2x$$

$\therefore$ G.S of (5) is $v = c_1\cos 2x + c_2\sin 2x + \sin 2x.$

$\therefore$ (3) $\equiv y = e^{x^2}(c_1\cos 2x + c_2\sin 2x) + \sin 2x$

which is the reqd g.s of (1)

H.W → $y'' - 2\tan x \cdot y' + 5y = 0.$

1997 → Make use of the transformation $y(x) = v(x)\sec(x)$ to obtain the solution of $y'' - 2y'\tan x + 5y_1 = 0$; $y(0) = 0, y'(0) = \sqrt{6}$

$y = \sin\sqrt{6}x \sec x$

③ To solve $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Qy = R(x)$ by changing the independent variable.

Given eqn is $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ —— ①

Let the independent variable 'x' be changed into another independent variable z.

where z is a function of 'x'

i.e let $z = f(x)$.

Now $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{dy}{dz}\frac{d^2z}{dx^2} + \frac{dz}{dx}\cdot\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \frac{dz}{dx}\cdot\frac{d}{dz}\left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \frac{dz}{dx}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right).$$

$$\frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2}$$

① $\Rightarrow \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} + P\cdot\frac{dy}{dz}\cdot\frac{dz}{dx} + Qy = R$

$$\Rightarrow \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} + \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\frac{dy}{dz} + Q\cdot y = R$$

$$\Rightarrow \frac{d^2y}{dz^2} + \left[\frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\right]\frac{dy}{dz} + \left(\frac{Q}{\left(\frac{dz}{dx}\right)^2}\right)y = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$\Rightarrow \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \quad \text{—— ②}$$

$y(x) = e^{ax}$, $y(x) = \frac{d^2y}{dz^2}$

where $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}$ ——— (3)

$Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2}$ ——— (4)

and $R_1 = \dfrac{R_1}{\left(\dfrac{dz}{dx}\right)^2}$ ——— (5)

Here $P_1, Q_1, R_1$ are functions of $x$. But these functions of $z$ by using $z = f(x)$

We now chose $z$ s.t $P_1 = 0$ and $Q_1 = \pm a^2$ (constant)

Case(i) If $P_1 = 0$ then (3) $\equiv \dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx} = 0$

$\Rightarrow \dfrac{d^2z}{dx^2} \Big/ \dfrac{dz}{dx} = -P$

$\Rightarrow \dfrac{\frac{d}{dx}\left(\frac{dz}{dx}\right)}{\frac{dz}{dx}} = -P$

$\Rightarrow \log\left(\dfrac{dz}{dx}\right) = -\int P\,dx$

$\Rightarrow \dfrac{dz}{dx} = e^{-\int P dx}$

$\Rightarrow \boxed{z = \int e^{-\int P dx}\,dx}$

Case (ii): If $Q_1 = \pm a^2$ (real constant) then from (4), we get

$\dfrac{Q}{(dz/dx)^2} = \pm a^2 \Rightarrow \pm Q = a^2\left(\dfrac{dz}{dx}\right)^2$

$\Rightarrow a\dfrac{dz}{dx} = \sqrt{\pm Q}$

$\Rightarrow a\int dz = \int\sqrt{\pm Q}\,dx$

(+ve or −ve sign is taken to make the expression under the radical sign +ve)

Problems :-

→ Solve the following:

(1) $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 4x^3y = 8x^3\sin x^2$ ——— (1)

Sol: $\dfrac{d^2y}{dx^2} - \dfrac{1}{x}\dfrac{dy}{dx} - 4x^2y = 8x^2\sin x^2$ ——— (2)

comparing ① with $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$.

$P = \frac{-1}{x}$; $Q = -4x^2$; $R = 8x^2\sin x^2$

Changing the independent variable from $x$ to $z$

where $z$ is a function of '$x$' i.e $z = f(x)$

∴ The given equation ① transformed into

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \quad \text{—— ②}$$

where $P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$;

$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$ ; $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$

Choosing $z$. s.t $\frac{Q}{\left(\frac{dz}{dx}\right)^2}$ = constant $= -1$ (say)

$\Rightarrow \left(\frac{dz}{dx}\right)^2 = 4x^2$

$\Rightarrow \frac{dz}{dx} = 2x$.

$\boxed{z = x^2}$

Now $P_1 = \frac{2 + \left(\frac{-1}{x}\right)(2x)}{4x^2} = 0$

$R_1 = \frac{8x^2\sin x^2}{4x^2} = 2\sin x^2 = 2\sin z$.

∴ ② $\equiv \frac{d^2y}{dz^2} + 0 - y = 2\sin z$

$\Rightarrow \frac{d^2y}{dz^2} - y = 2\sin z$ —— ③

$\Rightarrow (D^2 - 1)y = 2\sin z$ where $D = \frac{d}{dz}$

A.E is $D^2 - 1 = 0 \Rightarrow D = \pm 1$

$\therefore \boxed{C.F = c_1 e^{z} + c_2 e^{-z}}$

$P.I = 2 \cdot \frac{1}{D^2 - 1} \sin z$

$= 2 \cdot \frac{1}{-1-1} \sin z = -\sin z$

$\therefore$ G.S of (3) is $y = y_c + y_p$

$y = c_1 e^{z} + c_2 e^{-z} - \sin z$

$\therefore \boxed{y = c_1 e^{x^2} + c_2 e^{-x^2} - \sin x^2}$

which is the reqd g.s of (1).

→ $\cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2y \cos^3 x = \cos^5 x$

Sol: $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} - 2(\cos^2 x) y = 2\cos^4 x$ ——— (1)

$P = \tan x$ ; $Q = -2\cos^2 x$ ; $R = 2\cos^4 x$.

changing the independent variable $z$ from '$x$' to new independent variable $z$. where $z$ is a function of '$x$' i.e $z = f(x)$.

$\therefore$ The given equation is transformed into

$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$ ——— (2)

where $P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$ ; $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$ ; $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$

Let us choose $z$ s.t $\frac{-2\cos^2 x}{\left(\frac{dz}{dx}\right)^2}$ = constant $= -2$ (say)

$(c_1 \cosh\sqrt{2}z + c_2 \sinh\sqrt{2}z) + \frac{1}{4} + \frac{\sin^2 x}{2}$

$P_1 = 0$, $R_1 = 2\cos^2 x$
$= 2(1 - \sin^2 x)$

$= 2(1-z)$

$\therefore$ ② $\equiv \frac{d^2y}{dz^2} + 0 - 2y = 2(1-z)$

$\Rightarrow \frac{d^2y}{dz^2} - 2y = 2(1-z)$

Continue next soln.

H.W $\rightarrow y'' + \left(\frac{2}{x}\right)y' + \left(\frac{a^2}{x^4}\right)y = 0 \quad / \quad Q_1 = a^2$

H.W $\rightarrow x^4y'' + 2x^3y' + n^2y = 0 \quad / \quad Q_1 = n^2$

H.W $\rightarrow x^6y'' + 3x^5y' + a^2y = \frac{1}{x^2} \quad / \quad Q_1 = a^2$

$\rightarrow$ Transform the diff. equation $xy'' - y' + 4x^3y = x^5$ into $z$ as independent variable where $z = x^2$ and solve it.

Sol: Given that

$xy'' - y' + 4x^3y = x^5$ —— ①

and $z = x^2 \Rightarrow \frac{dz}{dx} = 2x$

Now $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$

$= \frac{dy}{dz}(2x)$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$= \frac{d}{dx}\left(\frac{dy}{dz} \cdot 2x\right)$

$= \frac{dy}{dz}(2) + 2x\frac{d}{dx}\left(\frac{dy}{dz}\right)$

$= 2\frac{dy}{dz} + 2x\frac{d}{dz}\left(\frac{dy}{dz}(2x)\right)$

$= 2\frac{dy}{dz} + (2x)^2\frac{d^2y}{dz^2}$

$\therefore$ ① $\equiv x\left[2\frac{dy}{dz} + 4x^2\frac{d^2y}{dz^2}\right] - 2x\frac{dy}{dz} + 4x^3y = x^5$

$\Rightarrow 4z\frac{d^2y}{dz^2} + 4z = z^2$

$\Rightarrow \frac{d^2y}{dz^2} + y = \frac{1}{4}z$ —— ②

Continue next soln

*→ An equation of the form

$$\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_n(x)y = Q(x)$$

where $P_1(x), P_2(x) \ldots P_n(x)$ & $Q(x)$ functions of $x$,

is called a linear diff. equation of order n.

This can be divided into two types:

(i) Homogenous linear diff. eqn.

(ii) Non-Homogenous linear diff. eqn.

→ If $Q(x)=0$ then ① is called homo.

→ If $Q(x)\neq 0$ then ① is called non-homo.

The wronskian condition:-

If $y=y_1(x), y=y_2(x), \ldots y=y_n(x)$ are solutions of $f(D)y=0$ then

$$W(y_1, y_2, \ldots y_n)(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

→ If $W(y_1, y_2, \ldots y_n)(x) \neq 0$ then the 'n' solutions are L.I - linearly Independent

→ $W(y_1, y_2, \ldots y_n)(x) = 0$ then the 'n' solutions are L.D. - linearly dependent

Note:- The $n^{th}$ order linear equation $f(D)y=0$ possesses 'n' distinct solutions which are LI

Ex:- $e^x, e^{2x}, e^{3x}$

let $y_1 = e^x, y_2 = e^{2x}, y_3 = e^{3x}$ then

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{2x} \end{vmatrix}$$

$$= e^{6x}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= 2e^{6x} \neq 0$$

$\therefore W \neq 0$

$\therefore$ The functions $e^x, e^{2x}, e^{3x}$ are L.I.

$\therefore y_1 = e^x, y_2 = e^{2x}, y_3 = e^{3x}$ are L.I solutions of $f(D)y=0$

Ex:-(2) $e^{2x}, e^{2x}, e^{-x}$

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^{2x} & e^{2x} & e^{-x} \\ 2e^{2x} & 2e^{2x} & -e^{-x} \\ 4e^{2x} & 4e^{2x} & e^{-x} \end{vmatrix}$$

$$= e^{3x}\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & 4 & 1 \end{vmatrix}$$

$$= e^{3x}[6-6+0]$$

$$= 0$$

$\therefore y = y_1, y_2, y_3$ are not L.I solns of $f(D)y=0$.

Note:- The $n^{th}$ order linear equation $f(D)y=0$ does not posses all solutions are distinct. Which are L.D.

## ④ Method of Variation of Parameters:-

To solve $\dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx} + Q(x)y = R(x)$ by the method of Variation of Parameters.

Given eqn is $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$

Its homogenous eqn is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad \text{——— ②}$$

Let $y_c = c_1 u(x) + c_2 v(x)$ be the g.s of (2) and hence it is c.f of (1)

Since $y = u(x)$, $y = v(x)$ are L.I solutions of (2)

$$\therefore \frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0$$

$$\frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv = 0 \quad \text{——— (3)}$$

Let $y_p = A\,u(x) + B\,v(x)$ be particular integral of (1) which is obtained from c.f of (1) by replacing $c_1$ & $c_2$ by A and B respectively

which are fns of 'x' Diff (4) w.r.t 'x', we get

$$\frac{dy_p}{dx} = A\frac{du}{dx} + u\frac{dA}{dx} + v\frac{dB}{dx} + B\frac{dv}{dx}$$

Now choosing A & B s.t

$$u\frac{dA}{dx} + v\frac{dB}{dx} = 0 \quad \text{——— (5)}$$

Then $\dfrac{dy_p}{dx} = A\dfrac{du}{dx} + B\dfrac{dv}{dx} \longrightarrow (6)$

Diff (6) w.r.t 'x', we get,

$$\frac{d^2y_p}{dx^2} = A\frac{d^2u}{dx^2} + \frac{dA}{dx}\cdot\frac{du}{dx} + B\frac{d^2v}{dx^2} + \frac{dB}{dx}\cdot\frac{dv}{dx} \quad \text{——— (7)}$$

(1) ≡

$$A\frac{d^2u}{dx^2} + \frac{dA}{dx}\frac{du}{dx} + B\frac{d^2v}{dx^2} + \frac{dB}{dx}\cdot\frac{dv}{dx} + P\left(A\frac{du}{dx} + B\frac{dv}{dx}\right) + Q(Au+Bv) = R$$

$$\Rightarrow A\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right] + B\left[\frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv\right] + \left(\frac{du}{dx}\frac{dA}{dx} + \frac{dv}{dx}\frac{dB}{dx}\right) = R$$

$$= A(0) + B(0) + \left(\frac{du}{dx}\frac{dA}{dx} + \frac{dv}{dx}\frac{dB}{dx}\right) = R \text{ (from (3))}$$

Solving (5) & (8)

$$\Rightarrow \frac{du}{dx}\frac{dA}{dx} + \frac{dv}{dx}\frac{dB}{dx} = R \quad \text{———} \ (8)$$

Solving (5) & (8) $\quad u\frac{dA}{dx} + v\cdot\frac{dB}{dx} - 0 = 0 \rightarrow (5).$

$$\frac{\frac{dA}{dx}}{vR} = \frac{\frac{dB}{dx}}{-uR} = \frac{-1}{uv' - vu'} \quad \text{where } u' = \frac{du}{dx}, \ v' = \frac{dv}{dx}$$

$$\Rightarrow \frac{dA}{dx} = \frac{-vR}{uv' - vu'};$$

$$\frac{dB}{dx} = \frac{uR}{uv' - vu'}$$

$$\Rightarrow A = -\int \frac{vR}{uv' - vu'}\,dx,$$

$$B = \int \frac{uR}{uv' - vu'}\,dx$$

After integration the constant is not added.

($\because$ A & B are involved in $y_p$)

$\therefore$ substituting the values of A & B in (4)

$\therefore$ The g.s of (1) is $y = y_c + y_p$

$$\Rightarrow y = c_1 u(x) + c_2 v(x) + A\,u(x) + B\,v(x)$$

Note:- (1) Since the form of $y_c$ & $y_p$ is the same.

But the constants which occur in $y_c$ are changed into ~~functions~~ of the independent variable 'x' in $y_p$.

For this reason the method of finding the P.I is called the method of variation of parameters.

(2) The above method can be extended to linear equations of order higher than the two.

(3) The above method is applicable to linear equations with constant coefficients and also variable co-efficients.

(4) W.K.T the given linear eqn of second order can be solved when part of C.F. is known.

$\frac{dy}{dx}+P(x)y=Q(x) \Rightarrow$ complimentary $\frac{dy}{dx}+P(x)y=0 \rightarrow y_c$ Find

$y_c = u(x)$ & $y_p = A(x)\cdot u(x)$ Substitute $y_p$ in $\frac{dy}{dx}+P(x)y=Q(x)$ to find $A(x)$

$\Downarrow$ $\boxed{\frac{du}{dx}+P(x)u=0}$   K. VENKANNA S. $y=y_c+y_p$ &



Therefore the above method is surely superior to the Variation of parameters.

Since this method requires a complete knowledge of the C.F. instead of one solution of it.

Hence the method of variation of parameters should be used only when specifically asked to solve by this method.

Working rule:

(1) write the given equation in the standard form $y''+Py'+Qy=R$.

(2) Find the solution of $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$.

Let it be $y_c = c_1u(x)+c_2v(x)$

(3) Let the P.I of the given eqn be $y_p = A(x)u+B(x)v$ where A & B are function of $x$.

(4) Find $\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv'-vu' = u\frac{dv}{dx}-v\frac{du}{dx}$.

(5) Find A and B by using, $A=\int\frac{-vR}{uv'-vu'}dx$ & $B=\int\frac{uR}{uv'-vu'}dx$

(6) The g.s of the given eqn is $y=y_c+y_p$

$\Rightarrow \boxed{y=(c_1u+c_2v)+(Au+Bv)}$

$\rightarrow$ Solve $(D^2+a^2)y = \tan ax$ by the method of variation of parameters.

Given that $(D^2+a^2)y = \tan ax$ —— (1)

A.E of (1) is $D^2+a^2=0$

$\Rightarrow D^2 = -a^2$

$\Rightarrow D = \pm ai$

$\therefore \boxed{y_c = c_1\cos ax + c_2\sin ax}$

Let $y_p = A\cos ax + B\sin ax$ be a P.I of (1)

where A & B fns of '$x$'.

then $u(x) = \cos x$; $v(x) = \sin ax$; $R(x) = \tan ax$.

Now $uv' - vu' = \cos ax\,(a\cos ax)$

$= +\sin ax\,(a\sin ax)$

$= a(\sin^2 ax + \cos^2 ax)$

$= a$

Now $A = \int \frac{-vR}{uv' - vu'}\,dx$

$= \frac{-1}{a}\int \left[\frac{1-\cos^2 ax}{\cos ax}\right] dx$

$= \frac{-1}{a}\int [\sec ax - \cos ax]\,dx$.

$= \frac{-1}{a}\left[\frac{\log|\sec ax + \tan ax|}{a} - \frac{\sin ax}{a}\right]$

$= -\frac{1}{a^2}\left[\log|\sec ax + \tan ax| - \sin ax\right]$

and $B = \int \frac{uR}{uv' - vu'}\,dx$

$= \frac{-1}{a^2}\cos ax$.

$\therefore y_p = \frac{1}{a^2}\left[\sin ax - \log|\sec x + \tan ax| - \cos ax\right]$

$\therefore$ The g.s of ① is $y = y_c + y_p$.

$\Rightarrow y = (c_1\cos ax + c_2\sin ax) + \frac{1}{a^2}\left[\sin ax - \log|\sec ax + \tan ax| - \cos ax\right]$

→ Solve $[(x-1)D^2 - xD + 1]y = (x-1)^2$ by the method of Variation of parameters.

Sol: Given eqn is

$$(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = (x-1)^2$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{x}{(x-1)}\frac{dy}{dx} + \frac{1}{(x-1)}y = (x-1)^2 \quad \text{—— ①}$$

Comparing ① with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$.

$P = \frac{-x}{x-1}$; $Q = \frac{1}{x-1}$; $R = (x-1)$.

Now the homo. eqn of (1) is

$$\frac{d^2y}{dx^2} - \frac{x}{x-1}\frac{dy}{dx} + \frac{1}{(x-1)}y = 0$$

Now $1+P+Q = 0$

$\therefore y = e^x$ is a part of C.F of (2).

Let $y_c = uv$ be the g.s of (2), where $u = e^x$

then $v$ is given by $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = 0$ ——— (3)

Now since $u = e^x \Rightarrow \frac{du}{dx} = e^x$

$$\therefore P + \frac{2}{u}\frac{du}{dx} = \frac{-x}{x-1} + \frac{2}{e^x}e^x$$

$$= \frac{-x}{x-1} + 2$$

$$= \frac{x-2}{x-1}$$

$$\therefore \frac{d^2v}{dx^2} + \left(\frac{x-2}{x-1}\right)\frac{dv}{dx} = 0 \quad \text{——— (4)}$$

Let $\frac{dv}{dx} = V$ then

$$\frac{dV}{dx} + \left(1 - \frac{1}{x-1}\right)V = 0.$$

$\Rightarrow \log V = \log(x-1) - x + \log c_1$

$\Rightarrow \log\left(\frac{V}{(x-1)c_1}\right) = -x$

$\Rightarrow V = c_1(x-1)e^{-x}$

$\Rightarrow \frac{dv}{dx} = c_1(x-1)e^{-x}$

$\Rightarrow v = -c_1 e^{-x}(x+1) + c_1 e^{-x} + c_2$

$$y_c = e^x\left[-c_1 e^{-x}(x+1) + c_1 e^{-x} + c_2\right]$$

$= -c_1(x-1) - c_1 + c_2 e^x$

$\therefore \boxed{y_c = c_1 x + c_2 e^x}$

Let $y_p = Au + Bv$ be a P.I of ① where $A$ & $B$ are fns of $x$

& $u = x$; $v = e^x$

Now $\begin{vmatrix} u & u' \\ v & v' \end{vmatrix} = uv' - vu'$

$= xe^x - e^x$

$= e^x(x-1)$

$\therefore A = \int \frac{-vR}{uv'-vu'} dx = \int -\frac{e^x \cancel{(x-1)}}{e^x \cancel{(x-1)}} dx$

$= -x$

and $B = \int \frac{uR}{uv'-vu'} dx = \int \frac{x \cancel{(x-1)}}{e^x \cancel{(x-1)}} dx$

$= \int x e^{-x} dx$

$= -e^{-x}(x+1)$.

$\therefore y_p = -x(x) - e^{-x}(x+1)e^x$

$= -x^2 - (x+1)$.

$= -(1+x+x^2)$

$\therefore$ G.S of ① is

$y = y_c + y_p$

$\Rightarrow \boxed{y = c_1 x + c_2 e^x - (1+x+x^2)}$

* Apply the method of variation of parameters to solve the following diff. eqns.

$\xrightarrow{1999}$ $y'' + n^2 y = \sec nx$ $\xrightarrow{2000}$ $x\frac{dy}{dx} - y = (x-1)\left(\frac{d^2y}{dx^2} - x + 1\right)$

$\xrightarrow{2001}$ $y'' + 4y = 4\tan 2x$ $\xrightarrow{2002}$ $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$ with $y(0) = 0$ and $\left(\frac{dy}{dx}\right)_{x=0} = 0$

$c_1 = +2$, $c_2 = +4$

$c_1x^3 + c_2x^2 - x^3\tan x - x^2[x\tan x + \log(\cos x)]$

$(c_1 + c_2 x)e^x + e^x(\sin x \cos x - x\cos x)$

$\xrightarrow{2003}$ $x^2y'' - 4xy' + 6y = x^4\sec^2 x$ $\xrightarrow{2005}$ $x^2y'' - 2xy' + 2y = x\log x$; $x > 0$.

$c_1x^2 + c_2x - \frac{x(\log x)^2}{2} - x\log x - x$

$\xrightarrow{2008}$ Use the method of variation of parameters to find the general solution of $x^2y'' - 4xy' + 6y = -x^4\sin x$

$\xrightarrow{\text{IFoS-2008}}$ $x^2y'' + xy' - y = x^2e^x$. $\longrightarrow$ $-\frac{c_1}{2x} + c_2 x - \frac{e^x}{2}\left[x - 2 + \frac{2}{x}\right] + \frac{xe^x}{2}$

# DIFFERENTIAL EQUTIONS OF FIRST ORDER BUT NOT OF FIRST DEGREE, Set-IV

If we denote $\frac{dy}{dx}$ by $p$ then the equation of the form $f(x,y,p)=0$ where $p$ is not of first degree, is called a differential equation of first order but not of first degree.

The most general form of a differential equation of first order and $n$th degree is $p^n + A_1 p^{n-1} + A_2 p^{n-2} + \dots + A_{n-1} p + A_n y = 0$

i.e $\left(\frac{dy}{dx}\right)^n + A_1\left(\frac{dy}{dx}\right)^{n-1} + \dots + A_{n-1}\left(\frac{dy}{dx}\right) + A_n y = 0$

where $A_1, A_2 \dots A_n$ are functions of $x$. Such equation can be divided into the following categories.

(i) solvable for $p$ (ii) solvable for $x$

(iii) solvable for $y$ (iv) Clairaut's equation.

## (i) Differential equations solvable for P:

Such equations can be resolved into linear factors of first degree. Let the given equation be

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + \dots + A_{n-1} p + A_n = 0 \quad \text{--- (1)}$$

Let it be resolved into linear factors to give

$$[p - f_1(x,y)][p - f_2(x,y)] \dots [p - f_n(x,y)] = 0$$

Then $p = f_1(x,y)$, $p = f_2(x,y)$ --- $p = f_n(x,y)$

These equations on integration give

$$F_1(x,y,c_1) = 0, \quad F_2(x,y,c_2) = 0, \dots\dots F_n(x,y,c_n) = 0$$

$\therefore$ The solution of (1) is

$$F_1(x,y,c_1) \cdot F_2(x,y,c_2) \dots F_n(x,y,c_n) = 0$$

But (1) being an equation of first order, its general solution must be contain one arbitrary constant.

Taking $c_1 = c_2 = c_3 = \cdots = c_n = c$

$\therefore$ The general solution (1) is

$$F_1(x,y,c),\ F_2(x,y,c) \cdots F_n(x,y,c) = 0$$

Problems:

$\rightarrow$ Solve the following differential equations.

(1) $(P-xy)(P-x^2)(P-y^2)=0$ ——— (I)

It is solving for P

$P-xy=0$ ——— ① ; $P=x^2$ ——— ② ; $P=y^2$ ——— ③

$\frac{dy}{dx}=xy$ ; $\frac{dy}{dx}=x^2$ ; $\frac{dy}{dx}=y^2$

$\Longrightarrow \log y = \frac{x^2}{2}+c$ ; $y=\frac{x^3}{3}+c$ ; $-\frac{1}{y}=x+c$

$\therefore$ General solution of (I) is

$$\left(\log y-\frac{x^2}{2}-c\right)\left(y-\frac{x^3}{3}-c\right)\left(-\frac{1}{y}+x+c\right)=0$$

$\rightarrow$ $(1+x)(xP+y+x)(P+2x)=0$

$\rightarrow$ $P^2-5P+6=0$ / $\Rightarrow (P-3)(P-2)=0$

$\rightarrow$ $x+yP^2=(1+xy)P$ / $\Rightarrow x+yP^2-P-xyP=0 \Rightarrow (x-P)-yP(x-P)=0$

$\rightarrow$ $4y^2P^2+2xy(3x+1)P+3x^3=0$

$\rightarrow$ $xy^2(P^2+2)=2Py^3+x^3$ / Add and sub $x^2yP$

$\rightarrow$ $yP^2+(x-y)P-x=0$

$\rightarrow$ $x^2P^2-2xyP+2y^2-x^2=0$

$\rightarrow$ $P^2x^2=y^2$ / $(Px)^2-y^2=0 \Rightarrow (Px-y)(Px+y)=0$

(ii) Differential Equations solvable for '$x$':

Let the given equation be solvable for '$x$' then it can be put in the form

$$x=f(y,P) \text{ ——— ①}$$

on diff. (1) w.r.t $y$, we get

$$\frac{dx}{dy}=F\left(y,P,\frac{dP}{dy}\right) \quad (\text{or}) \quad \frac{1}{P}=F\left(y,P,\frac{dP}{dy}\right) \text{ ——— ②}$$

Let the solution of ② be $\phi(y,p,c)=0$ —— ③

Eliminating $p$ b/w ① & ③ is given the solution of the given equation.

If it is not possible to eliminate '$p$' then the values of '$x$' and '$y$' interms of $p$ in the form $x=f_1(p,c)$ & $y=f_2(p,c)$ together give the solution.

→ Solve the following differential equations

(1) $xp^3 = a+bp$

Sol'n: Given that $xp^3 = a+bp$ —— ①

It is solving for '$x$'

we have $x = \frac{a}{p^3}+\frac{bp}{p^3}$

$$\Rightarrow x = \frac{a}{p^3}+\frac{b}{p^2} \quad \text{—— ②}$$

Diff ② w.r.t. '$y$' we get

$$\frac{dx}{dy} = \left(\frac{-3a}{p^4}-\frac{2b}{p^3}\right)\frac{dp}{dy}$$

$$\frac{1}{p} = \frac{1}{p}\left(\frac{-3a}{p^3}-\frac{2b}{p^2}\right)\frac{dp}{dy} \qquad \left(\because \frac{dx}{dy}=\frac{1}{p}\right)$$

$$\Rightarrow dy = \left(\frac{-3a}{p^3}-\frac{2b}{p^2}\right)dp$$

$$\Rightarrow y = \frac{3a}{2p^2}+\frac{2b}{p}+c \quad \text{—— ③}$$

It is not possible to eliminate '$p$' from ② & ③.

∴ General solution of ① is $x = \frac{a}{p^3}+\frac{b}{p^2}$ &

$$y = \frac{3a}{2p^2}+\frac{2b}{p}+c$$

→ $p^3-4xyp+8y^2=0$ —— ①

Sol'n: $4xyp = p^3+8y^2$

$$x = \frac{p^2}{4y}+\frac{2y}{p} \quad \text{—— ②}$$

Diff. w.r.t 'y', we get,

$$\frac{dx}{dy} = \frac{1}{4}\frac{\left[y(2p)\frac{dp}{dy} - p^2\right]}{y^2} + \frac{2\left[p - y\frac{dp}{dy}\right]}{p^2}$$

$$\Rightarrow \frac{1}{p} = \frac{p}{2y}\frac{dp}{dy} - \frac{p^2}{4y^2} + \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} = \left(\frac{p}{2y} - \frac{2y}{p^2}\right)\frac{dp}{dy} + \left(\frac{2}{p} - \frac{p^2}{4y^2}\right)$$

$$\Rightarrow \left(\frac{p}{2y} - \frac{2y}{p^2}\right)\frac{dp}{dy} + \frac{1}{p} - \frac{p^2}{4y^2} = 0$$

$$\Rightarrow \left(\frac{p}{2y} - \frac{2y}{p^2}\right)\frac{dp}{dy} + \frac{p}{2y}\left(\frac{2y}{p^2} - \frac{p}{2y}\right) = 0$$

$$\Rightarrow \left(\frac{p}{2y} - \frac{2y}{p^2}\right)\left(\frac{dp}{dy} - \frac{p}{2y}\right) = 0$$

Omitting the first factor which leads to a singular solution, we get.

$$\frac{dp}{dy} - \frac{p}{2y} = 0$$

$$\Rightarrow \frac{dp}{p} = \frac{1}{2}\,\frac{1}{y}\,dy$$

$$\Rightarrow \log p = \frac{1}{2}\log y + \log c$$

$$\Rightarrow p = y^{1/2}c \quad \text{—— (3)}$$

Now eliminating $p$ b/w (2) & (3), we get

$$x = \frac{yc^2}{4y} + \frac{2y}{y^{1/2}c}$$

$$\Rightarrow x = \frac{c^2}{4} + \frac{2y^{1/2}}{c}$$

which is the required general solution of (1).

Note: The factor which does not involve a derivative of $p$ w.r.t '$x$' or '$y$' will be omitted. Such factor always lead to singular solutions.

→ $x = p^2$

→ $ap^2 + py - x = 0$

→ $y^2\log y = xyp + p^2$

'008 → solve the equation $y - 2xp + yp^2 = 0$ where $p = \frac{dy}{dx}$.

## * Differential Equations solvable for 'y':

Let the given equation be $y = f(x, p)$ ——— ①

Diff. w.r.t 'x' and writing $\frac{dy}{dx} = p$,

we get an equation of the form

$$p = F\left(x, p, \frac{dp}{dx}\right).$$

This is a differential equation in $p$ & $x$ and we get its solution in the form $\phi(x, p, c) = 0$.

Eliminating $p$ b/w (1) & ② we get the required solution.

If it is not possible to eliminate 'p' then ① & ② can be put in the form $x = f_1(p, c)$,

$y = f_2(p, c)$ and

these two equations together constitute the solution.

Problems:

→ Solve the following Differential Equations:

(1) $y = 3x + \log p$ ——— ①

It is solving for 'y'

Now diff. w.r.t 'x' we get

$$\frac{dy}{dx} = 3 + \frac{1}{p}\frac{dp}{dx}$$

$$\Rightarrow p = 3 + \frac{1}{p}\frac{dp}{dx} \qquad \left(\because \frac{dy}{dx} = p\right)$$

$$\Rightarrow \frac{1}{p}\frac{dp}{dx} = p - 3$$

$$\Rightarrow \frac{dp}{p(p-3)} = dx$$

$$\Rightarrow \frac{1}{3}\left[\frac{1}{p-3} - \frac{1}{p}\right]dp = dx$$

$$\Rightarrow \left[\log(p-3) - \log p\right] = 3x + c$$

$$\Rightarrow \log\left(\frac{p-3}{p}\right) = 3x + c$$

$$\Rightarrow \frac{p-3}{p} = e^{3x+c}$$

$$\Rightarrow 1 - \frac{3}{p} = e^{3x+c}$$

$$\Rightarrow 1 - e^{3x}.e^{c} = \frac{3}{p}$$

$$\Rightarrow p = \frac{3}{1-e^{3x}e^{c}} \quad \text{——— (2)}$$

Eliminating p from (1) & (2) we get,

$$y = 3x + \log\left[\frac{3}{1-e^{3x}e^{c}}\right]$$

which is required general solution of (1).

→ $x^2 + p^2 x = yp$

→ $x = y - p^2$

~~→~~ $p^3 - p(y+3) + x = 0$

→ $y = 2p + 3p^2$

## Clairaut's Equation:

The differential Equation of the form $y = xp + f(p)$ is known as Clairauts equation.

This equation is solved by considering the equation $y = f(x,p)$, solvable for y.

## Solution of Clairauts equation:

Let the given equation be

$$y = xp + f(p) \quad \text{——— (1)}$$

Diff w.r.t 'x' we get

$$\frac{dy}{dx} = x\frac{dp}{dx} + p + f'(p)\frac{dp}{dx}$$

$$\Rightarrow p = x\frac{dp}{dx} + p + f'(p)\frac{dp}{dx}$$

$$\Rightarrow (x + f'(p))\frac{dp}{dx} = 0.$$

$\Rightarrow \frac{dP}{dx} = 0$ ($x + f'(P)$ is discarded)

$\Rightarrow dP = 0 \Rightarrow \boxed{P = c}$

$\boxed{① \equiv y = xc + f(c)}$

which is the required general solution of ①.

**Working Rule:**

Given equation can be put in the form $y = xp + f(P)$ —— ①

In order to find its solution

replace 'p' by 'c'.

∴ The general solution of ① is $\boxed{y = xc + f(c)}$

**Problems:**

→ Solve $y = x\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2$.

Sol'n: Given that $y = x\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2$ —— ①

Let $\frac{dy}{dx} = p$ then

$y = xp + p^2$ —— ②

Clearly which is in Clairaut's form

∴ Replacing p by c in ②, we get

$y = xc + c^2$ which is the required general solution of ①.

→ Solve the following differential Equations:

(1) $P = \log(Px - y)$

(2) $y = xp + \frac{a}{p}$

(3) $\sin(y - xp) = P$ (or) $\sin px \cos y = \cos px \sin y + p$

(4) $(y - xp)(P - 1) = P$

(5) $(xp - y)^2 = p^2 - 1 \Rightarrow xp - y = \pm\sqrt{p^2 - 1}$.

## Equations reducible to Clairaut's form:

**Form I :**

$$y^2 = pxy + f\left(\frac{py}{x}\right) \quad \text{——— (1)}$$

Put $x^2 = X$ ; $y^2 = Y$

$2x\,dx = dX$ ; $2y\,dy = dY$

$\Rightarrow \frac{dY}{dX} = \frac{y}{x}\frac{dy}{dx} \Rightarrow \boxed{P = \frac{y}{x}p}$

$\Rightarrow \boxed{p = \frac{x}{y}P}$

$\therefore (1) \equiv Y = \frac{x}{y}P.(xy) + f\left(\frac{x}{y}P\frac{y}{x}\right)$

$\Rightarrow Y = x^2P + f(P)$

$\Rightarrow \boxed{Y = XP + f(P)}$ ——— (2)

clearly which is Clairaut's form

The general solution of (2) is $Y = cX + f(c)$

$\Rightarrow y^2 = cx^2 + f(c)$ is the general solution of (1).

→ solve the following differential equations by using the transformations $x^2 = X$, $y^2 = Y$.

1996 (1) $x^2(y - px) = yp^2 \Rightarrow x^2y^2 - x^3py = y^2p^2$

$\Rightarrow y^2 = pxy + \left(\frac{py}{x}\right)^2$

(2) $(px - y)(py + x) = h^2p$

(3) $(px - y)(x - yp) = 2p$.

*** Form II :**

If $e^{by}(a - bp) = f(pe^{by - ax})$

then put $e^{ax} = X$ , $e^{by} = Y$

**Problems :**

→ solve $e^{3x}(p-1) + p^3e^{2y} = 0$ ——— (1)

Sol'n : $e^{3x}(1-p) = p^3e^{2y}$

$\Rightarrow (1-p) = p^3e^{2y-3x}$

$\Rightarrow e^y(1-p) = p^3 e^{3y-3x}$

$\Rightarrow e^y(1-p) = (pe^{y-x})^3$

Clearly which is in the form of

$e^{by}(a-bp) = f(pb\,e^{by-ax})$, where $b=1, a=1$

Let $e^x = X$; $e^y = Y$

$e^x dx = dX$; $e^y dy = dY$

$\Rightarrow \frac{dY}{dX} = \frac{e^y}{e^x}\frac{dy}{dx}$

$\Rightarrow P = \frac{e^y}{e^x} p$

$\Rightarrow \boxed{p = \frac{e^x}{e^y} P}$

$\therefore$ ① $\equiv X^3\left(\frac{e^x}{e^y}P - 1\right) + \left(\frac{e^x}{e^y}P\right)^3 (Y)^2 = 0$

$\Rightarrow X^3\left(\frac{X}{Y}P-1\right) + \left(\frac{X}{Y}P\right)^3 Y^2 = 0$

$\Rightarrow XP - Y + P^3 = 0$

$\Rightarrow \boxed{Y = XP + P^3}$

which is in Clairaut's form.

$\therefore$ It's general solution is $Y = Xc + c^3$

$\Rightarrow e^y = e^x c + c^3$ is general solution of ①

* Solve the following differential equations by using the mentioned transformations.

| Differential Equation | Transformation. |
|---|---|
| $y = 2xp + y^2p^3$ (or) $y = 2px + y^{n-1}p^n$ | $y^2 = Y \Rightarrow 2y\,dy = dY$ $\Rightarrow 2y\frac{dy}{dx} = \frac{dY}{dx}$ $\Rightarrow 2yp = P$ where $p = \frac{dy}{dx}, P = \frac{dY}{dx}$ |
| $y + px = x^4p^2$ | $\frac{1}{x} = X$ |
| $y = 3px + 6y^2p^2$ | $y^3 = Y$ |
| $\cos^2 y\, p^2 + \sin x \cos x \cos y\, p - \sin y \cos^2 x = 0$ | $\sin x = X$ ; $\sin y = Y$ |
| $(xp - y)^2 = a(1+p^2)(x^2+y^2)^{3/2}$ | $x = r\cos\theta$ $y = r\sin\theta$ |
| $(xp - y)^2 = (x^2 - y^2)\sin^{-1}(y/x)$ | $y = vx$ |
| $(x^2+y^2)(1+p)^2 - 2(x+y)(1+p)(x+py) + (x+py)^2 = 0$ | $x + y = u$ $x^2 + y^2 = v$ |
| $x^2p^2 + 2xyp + y^2(1+p) = 0$ | $y = Y$ ; $xy = Y$ $y = u$ ; $xy = v$ |
| $(px^2 + y^2)(px + y) = (p+1)^2$ | $x + y = x$ ; $\cancel{y = Y}$ $xy = Y$. $x + y = u$ ; $xy = v$ |
| $xp^2 - 2yp + x + 2y = 0$ | $x^2 = X$, $y - x = Y$ |
| $y^2(y - xp) = x^4p^2$ | $\frac{1}{x} = X$ ; $\frac{1}{y} = Y$. |

## * Singular Solutions:

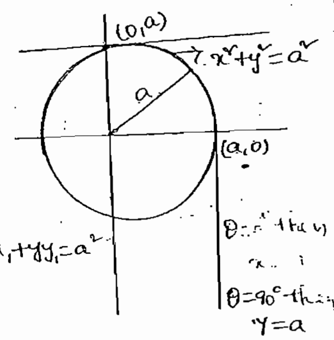
Def: A solution of a differential equation which is not derived from general solution by giving the particular values to the arbitrary constants, is called a singular solution. The singular solution does not involve any arbitrary constant.

## Envelopes:

→ Consider the equation $x\cos\theta + y\sin\theta = a$, where '$a$' is constant.

For different values of $\theta$, the equation represents a family of straight lines touching the circle $x^2+y^2=a^2$.

Here $\theta$ is the parametre of the family of straight lines $x\cos\theta + y\sin\theta = a$.



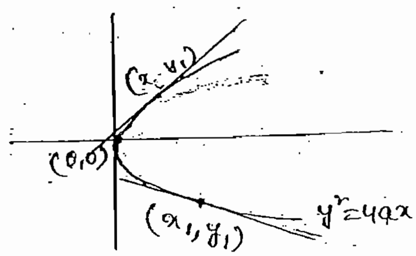

By the above example, the circle which is touched by a family of straight lines, is called the envelope of the family of straight lines.

i.e. the curve E which is touched by a family of curves 'C' is called the envelope of the family of curves 'C'.

## Equation of Tangent:

The equation of tangent at $(x_1,y_1)$ to the parabola $y^2=4ax$ is

$$yy_1 = 2a(x+x_1) \quad \text{——(i)}$$




## * Equation of Tangent in terms of slope (m):

The equation of tangent (i) is

$$y = \left(\frac{2a}{y_1}\right)x + \left(\frac{2ax_1}{y_1}\right) \quad \text{——(ii)}$$

Let the slope of tangent at $(x_1,y_1)$ be 'm'

Now the eqn of the tangent at $P(x_1,y_1)$ on the parabola $y^2=4ax$ —①

Diff. ① w.r.t 'x'

$2yy' = 4a \Rightarrow \frac{dy}{dx}\Big/_{(x_1,y_1)} = \frac{4a}{2y_1}$

∴ The eqn of tangent at $(x_1,y_1)$ on the parabola is $(y-y_1) = \frac{2a}{y_1}(x-x_1)$

$y^2=4x$; $a=1$

$yy_1 = 2(x+x_1)$

$(x_1,y_1)=(1,2)$

∴ $2y = 2(x+1)$

$\Rightarrow x-y+1=0$

$x=1, y=2$

$(x_1,y_1)=(0,0)$

$\Rightarrow x=0$

$\therefore \frac{2a}{y_1} = m$

$\Rightarrow \boxed{y_1 = \frac{2a}{m}}$

Since $(x_1, y_1)$ lies on parabola $y^2 = 4ax$

$\Rightarrow y_1^2 = 4ax_1 \Rightarrow \frac{4a^2}{m^2} = 4ax_1$

$\Rightarrow \boxed{x_1 = \frac{a}{m^2}}$

$\therefore (x_1, y_1) = \left(\frac{a}{m^2}, \frac{2a}{m}\right)$ is the point of contact.

$\therefore$ The equation of tangent at $(x_1, y_1) = \left(\frac{a}{m^2}, \frac{2a}{m}\right)$

is $\boxed{y = mx + \frac{a}{m}}$

Note: By the definition of envelope, $y^2 = 4ax$ is the envelope of the straight lines $y = mx + \frac{a}{m}$; m being parameter.

→ Let us consider the differential equation.

$$y = px + \frac{a}{p} \; ; p = \frac{dy}{dx} \quad \text{---} (1)$$

Clearly which is in Clairaut's form

$\therefore$ It's general solution is $y = mx + \frac{a}{m}$; — (2)

'm' is the parameter.

We know that the equation (2) is the tangent to the Parabola $y^2 = 4ax$ — (3)

clearly (3) is the envelope of (2)

Now we show that $y^2 = 4ax$ is also solution of (1)

Diff (3) w.r.t 'x', we get

$2yy' = 4a$

$\therefore \boxed{y' = \frac{2a}{y}}$ &

The whole parabola (3) Covered by (2) for different values of 'm'

$\therefore$ (3) is the soln of (1)

from ③, $\boxed{x = \frac{y^2}{4a}}$

$\therefore$ ① $\equiv y = \left(\frac{2a}{y}\right)\left(\frac{y^2}{4a}\right) + a\left(\frac{y}{2a}\right)$

$\Rightarrow y = y$ (identity)

$\therefore$ ③ is the solution of ①.

→ The equation of the envelope of the family of Curves given by the general solution of a differential equation is known as the Singular solution. Such a solution does not contain any arbitrary constant. and is not a particular case of the general solution.

It is some times possible to reduce this solution from the general solution by giving the particular values to the arbitrary constants. In such a case the singular solution is also called particular solution.

→ Let $f(x,y,p) = 0$ be the differential equation whose solution is $\phi(x,y,c) = 0$, then p-discriminant is obtained by eliminating p between $f(x,y,p) = 0$ & $\frac{\partial f}{\partial p} = 0$.

If ~~p~~ ~~pis~~ ~~en~~ $f(x,y,p)$ is quadratic eqn $\Rightarrow b^2 - 4ac = 0$.

Also c-discriminant is obtained by eliminating c between $\phi(x,y,c) = 0$ & $\frac{\partial \phi}{\partial c} = 0$.

→ If $E(x,y) = 0$ is a singular solution (envelope) of the differential equation $f(x,y,p) = 0$. whose general solution.

is $\phi(x,y,c)=0$ then $E(x,y)$ is a factor of both the discriminants (i.e. P & C) and $E(x,y)=0$ must satisfy the differential equation $f(x,y,P)=0$.

Ex: show that $x=0$ is the singular solution of

$4xp^2=(3x-a)^2$

Sol$^n$: The given differential equation is

$$4xp^2-(3x-a)^2=0 \quad \text{——— (1)}$$

general solution is $(y+c)^2=x(x-a)^2$ ——— (2)

P-discriminants

$$(0)^2-4\cdot 4x\left[(3x-a)^2\right]=0$$

$$\Rightarrow x(3x-a)^2=0 \quad \text{——— (3)} \qquad (\because b^2-4ac=0)$$

C-discriminant:

$$(2)\equiv c^2+2yc+y^2-x(x-a)^2=0$$

$$\Rightarrow (2y)^2-4(1)\left[y^2-x(x-a)^2\right]=0$$

$$\Rightarrow 4y^2-4y^2+x(x-a)^2=0$$

$$\Rightarrow x(x-a)^2=0 \quad \text{——— (4)}$$

$\therefore x=0$ is the factor of both the discriminants.

and it must be satisfy the given differential Equation.

Since from (1), $4x-\dfrac{(3x-a)^2}{p^2}=0$

which is satisfied by $x=0$

because $\dfrac{dx}{dy}=0$ i.e. $\dfrac{1}{p}=0$

$\therefore x=0$ is the singular solution of (1).

→ Each discriminant may have other factor which correspond to other loci associated with the general solution of the given differential Equation. Generally the equations of these loci donot satisfy the differential equation, they are known as extraneous loci.

These are three types of Extraneous loci

(1) Tac-locus (2) Node-locus and (3) Cusp-locus.

Methods for finding the singular solution:

→ To find the singular solution of a diff. equation $f(x,y,P)=0$

(1) Find its general solution $\phi(x,y,c)=0$.

(2) Find P-discriminant

(3) Find C-discriminant

* Now P-discriminant equated to zero may include as a factor.

(1) Envelope i.e. singular solution once (E)

(2) cusp-locus once (C)

(3) Tac-locus twice $(T^2)$

i.e. P-discriminant $= ET^2C$

* C-discriminant equated to zero may include as a factor:

(1) Envelope i.e. singular solution

(2) cusp-locus thrice $(C^3)$

(3) Node-locus twice $(N^2)$

i.e. C-discriminant $= EN^2C^3$

Problems: SetA

(when equations are solvable for P)

Obtain the complete primitive (i.e. g.s) and singular solution of the following equations. Explaing the geometrical significance of the irrelevent factors that present themselves.

(1) $xp^2 = (x-a)^2$

Sol'n : Given that $xp^2 = (x-a)^2$ —— ①

It is solvable for 'p'.

$\therefore [\sqrt{x}\,p - (x-a)][\sqrt{x}\,p + (x-a)] = 0$ —— ②

$\Rightarrow \sqrt{x}\,p = x-a; \quad \sqrt{x}\,p = a-x$

$\Rightarrow \frac{dy}{dx} = \sqrt{x} - a x^{-1/2}$ | $\Rightarrow \frac{dy}{dx} = a x^{-1/2} - \sqrt{x}$

$\Rightarrow dy = (\sqrt{x} - a x^{-1/2})dx$ | $\Rightarrow dy = (a x^{-1/2} - \sqrt{x})dx$

$\Rightarrow y = \frac{2}{3}x^{3/2} - 2a x^{1/2} + c$ | $\Rightarrow y = 2a x^{1/2} - \frac{2}{3}x^{3/2} + c$

$\therefore \left(y - \frac{2}{3}x^{3/2} + 2a x^{1/2} - c\right)\left(y - 2a x^{1/2} + \frac{2}{3}x^{3/2} - c\right) = 0$

$\Rightarrow (y-c)^2 - \left(\frac{2}{3}x^{3/2} - 2a x^{1/2}\right)^2 = 0$

$\Rightarrow (y-c)^2 = \frac{4}{9}x(x-3a)^2$ —— ③

which is the required general solution of ①

Now P-discriminant :

$0 - 4x[-(x-a)^2] = 0$

$\Rightarrow 4x(x-a)^2 = 0$

$\Rightarrow x(x-a)^2 = 0$ —— ④

C-discriminant :

③ $\equiv y^2 + c^2 - 2yc = \frac{4}{9}x(x-3a)^2$

$\Rightarrow c^2 + (-2y)c + y^2 - \frac{4}{9}x(x-3a)^2 = 0$

$\Rightarrow (-2y)^2 - 4(1)\left[y^2 - \frac{4}{9}x(x-3a)^2\right] = 0 \quad (\because b^2 - 4ac = 0)$

$\Rightarrow 4y^2 - 4\left[y^2 - \frac{4}{9}x(x-3a)^2\right] = 0$

$\Rightarrow \frac{4}{9}x(x-3a)^2 = 0$

$\Rightarrow x(x-3a)^2 = 0$ —— ⑤

$\therefore x=0$ is the factor of both the discriminants (i.e. P&C)

and $x=0$ satisfies the given diff. equation.

$\therefore x=0$ is a singular solution.

Now $x-a=0$ is a tac-locus because it appears twice in the P-discriminant relation ④,

it does not occur in the C-discriminant relation ⑤ and does not satisfy the differential Equation ①

Now $x-3a=0$ is a node-locus because it appears twice in the C-discriminant relation ⑤, it does not occur in the P-discriminant relation ④ and does not satisfy the diff. equation ①.

(2) $4p^2x(x-a)(x-b) = [3x^2-2x(a+b)+ab]^2$

Given equation is

$$4p^2x(x-a)(x-b)^2 = [3x^2-2x(a+b)+ab]^2 \quad \text{--- } ①$$

Its general solution

$$(y+c)^2 = x(x-a)(x-b)$$

$$\Rightarrow c^2+2cy+y^2-x(x-a)(x-b)=0 \quad \text{--- } ②$$

Now P-discriminant:

$$0-4(4x)(x-a)(x-b)[-(3x^2-2x(a+b)+ab]^2=0$$

$$\Rightarrow x(x-a)(x-b)[3x^2-2x(a+b)+ab]^2=0 \quad \text{--- } ③$$

C-discriminant:

$$4y^2-4\cdot1[y^2-x(x-a)(x-b)]=0$$

$$\Rightarrow x(x-a)(x-b)=0$$

Here $x=0$ appears once in both the discriminants and it satisfies the equation ①.

i.e. $4x(x-a)(x-b)-\dfrac{[3x^2-2x(a+b)+ab)]^2}{p^2}=0$ $\left[\because x=0 \Rightarrow \dfrac{dx}{dy}=0 = \dfrac{1}{p}=0\right]$

$\therefore x=0$ is the singular solution.

Similarly $(x-a)=0$ and $(x-b)=0$ are also singular solutions.

Now $3x^2-2x(a+b)+ab=0$ is a tac-locus because it appears twice in the P-discriminant.

Again solving it for 'x' we get

$$x=\frac{2(a+b)\pm\sqrt{4(a+b)^2-12ab}}{6}$$

$$\Rightarrow 3x=(a+b)\pm(a^2-ab+b^2)^{1/2} \quad \text{——— (5)}$$

The above mentioned 'tac-locus' factors are given by (5)

$\therefore$ There are two tac-loci given by (5)

Set-B

(When equations are solvable for 'x')

→ Solve and examine for singular solutions.

$$p^3-4xyp+8y^2=0 \quad \text{——— (1)}$$

Sol'n: it is solvable for 'x',

$$4xyp=p^3+8y^2$$

$$\Rightarrow x=\frac{p^2}{4y}+\frac{2y}{p} \quad \text{——— (2)}$$

Diff. w.r.t 'y' we get so on

It's g.s. is $\boxed{y=c(c-x)^2}$ ——— (3)

Now P-discriminant:

Diff. (1) partially w.r.t 'p', we get

$$3p^2-4xy=0$$

$$\Rightarrow p^2=\frac{4xy}{3} \quad \text{—— (4)}$$

Now eliminating p from (1) & (4)

for this (1) $\equiv 8y^2=p(4xy-p^2)$

$$\Rightarrow 64y^4=p^2(4xy-p^2)^2$$

$$\Rightarrow 64y^4=\frac{4xy}{3}\left(4xy-\frac{4xy}{3}\right)^2 \quad (\text{from (4)})$$

$$\Rightarrow 64y^4=\frac{64xy}{3}\left(\frac{2xy}{3}\right)^2$$

Set E:
(Equations Reducible to clairaut's form)

→ Reduce the differential equation $(px-y)(x-py)=2p$. to clairaut's form by the substitution $x^2=u$ and $y^2=v$ and find its complete primitive and its singular solution, if any

Sol'n: Given that $(px-y)(x-py)=2p$ —— ①

It's g.s is

$$y = cx - \frac{2c}{1-c} \quad \text{(by previous methods)}$$

(or)

$$x^2c^2-(x^2+y^2-2)c+y^2=0 \text{ —— ②}$$

$$① \equiv xyp^2-(x^2+y^2-2)p+xy=0 \text{ —— ③}$$

Now p-discriminant is:

$$(x^2+y^2-2)^2-(2xy)^2=0 \text{ —— ④}$$

$$\Rightarrow (x^2+y^2-2-2xy)(x^2+y^2-2+2xy)=0$$

$$\Rightarrow [(x-y)^2-(\sqrt{2})^2][(x+y)^2-(\sqrt{2})^2]=0$$

$$\Rightarrow (x-y+\sqrt{2})(x-y-\sqrt{2})(x+y+\sqrt{2})(x+y-\sqrt{2})=0$$

Now from ②, c-discriminant is

$$(x^2+y^2-2)^2-(2xy)^2=0$$

which is same as ④

∴ It also reduces to ④

∴ p-and c-discriminant relations are coincident here.

$x-y+\sqrt{2}=0$ appears in both the discriminant and satisfies the given diff. equation and hence it is a singular solution.

Similarly,

$x-y-\sqrt{2}=0$ , $x+y+\sqrt{2}=0$ and

$x+y-\sqrt{2}=0$ are also singular solutions.

Reduce the equation $xyp^2-(x^2+y^2-1)p+xy=0$ to clairaut's form by the substitution $x^2=u$ and $y^2=v$. Hence show that the equation represents a family of conics touching the four sides of a square.

Sol'n: Given that $xyp^2-(x^2+y^2-1)p+xy=0$ —— ①

It's g.s is $c^2x^2-c(x^2+y^2-1)+y^2=0$ —— ② (by using previous methods)

which represents a family of conics.

from ①: the P-discriminant relation is

$$(x^2+y^2-1)^2-4x^2y^2=0 \quad \text{—— ③}$$

from ②: the C-discriminant relation is

$$(x^2+y^2-1)^2-4x^2y^2=0 \quad \text{—— ④}$$

∴ from ③ & ④, $(x^2+y^2-1)^2-4x^2y^2=0$ —— ⑤

must be singular solution, because it is present once in both the discriminants.

Again from ⑤, we have

$$(x+y+1)(x+y-1)(x-y+1)(x-y-1)=0$$

$\Rightarrow x+y+1=0$ , $x+y-1=0$ , $x-y+1=0$ and $x-y-1=0$

are four singular solutions (envelops).

Thus the given diff. equation represents a family of conics given by ② which are touched by the four lines (envelops) mentioned above and the four lines form the four sides of a square.

$$\Rightarrow y^4 = \frac{4x^3y^3}{27}$$

$$\Rightarrow 27y^4 = 4x^3y^3$$

$$\Rightarrow y^3\left[27y - 4x^3\right] = 0$$

$$\Rightarrow y^2 \cdot y\left[27y - 4x^3\right] = 0 \quad \text{---} \ (5)$$

C-discriminant:

Diff (3) partially w.r.t. 'c' we get

$$0 = (c-x)^2 + 2c(c-x)$$

$$\Rightarrow (c-x)(3c-x) = 0$$

$$\Rightarrow c = x \ \& \ c = x/3$$

If $c = x$ then (3) $\equiv y = 0$ —— (6)

If $c = x/3$ then (3) $\equiv y = \frac{4x^3}{27}$ —— (7)

from (6) & (7) we have

$$y\left(y - \frac{4x^3}{27}\right) = 0$$

$$\Rightarrow y(27y - 4x^3) = 0 \quad \text{---} \ (8).$$

$\therefore y = 0$ & $27y - 4x^3 = 0$ are the singular solutions.

Because they both appear once in (5) & (8) and satisfying the given differential equation.

→ Find the solution of the differential equation $y = 2xp - yp^2$ Also find the singular solution.

Soln:

(when equations are solvable for y)

→ Solve the differential equation $(8p^3 - 27)x = 12p^2y$ and investigate whether a singular solution exists.

* Examine the following equations for singular solution and extraneous loci if any

000 → $y = x - 2ap + ap^2$ ..

001 → $xp^2 - 2yp + \cancel{a a = 0}$

→ $ap^2 - 2yp + 4x = 0$

Set-D

(when equations are in clairaut's form)

→ Find the complete solution & singular solution of

$$y = px + \sqrt{b^2 + a^2p^2}$$

Sol'n: Given that $y = px + \sqrt{b^2 + a^2p^2}$ ——— ①

Clearly which is in clairaut's form

∴ It's g.s is

$$y = cx + \sqrt{b^2 + a^2c^2} \quad \text{———} \quad ②$$

Now from ① & ② both P & C – discriminant:

For this (1)

$$(y - px)^2 = b^2 + a^2p^2$$

$$\Rightarrow (x^2 - a^2)p^2 - 2xyp + y^2 - b^2 = 0$$

$$\Rightarrow 4x^2y^2 - 4[x^2 - a^2][y^2 - b^2] = 0$$

$$\Rightarrow x^2y^2 - (x^2y^2 - x^2b^2 - a^2y^2 + a^2b^2) = 0$$

$$\Rightarrow x^2b^2 + a^2y^2 - a^2b^2 = 0$$

which is the P– & C – discriminant.

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which must be the singular solution because it is present in both the discriminants and satisfies the given differential equation.

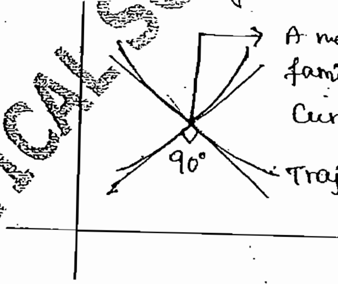
Note: In case of clairaut's equation P-and C– discriminants are always identical.

# Orthogonal Trajectories

**Definition:**

→ A curve which cuts every ~~member~~ of ~~a~~ given family of curves according to a given law, is called a Trajectory of the given family of curves.

→ A trajectory of a family of curves is called an orthogonal trajectory of the family if it cuts every member of the family at right angle.

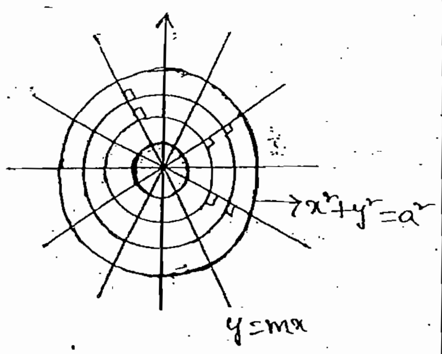



→ A trajectory of a family of curves called an oblique trajectory of the family if it cuts every member of the given family of curves at an angle $\alpha \neq 90°$.

Ex: Consider the two family of curves $y = mx$ & $x^2+y^2=a^2$ where m & a are parameters.

then $y=mx$ is an orthogonal trajectory of the family of circles $x^2+y^2=a^2$.

Since every line (i.e. $y=mx$) passing through the origin of coordinates is an orthogonal trajectory of the family of the concentric circles (i.e. $x^2+y^2=a^2$)




## Working Rule for Finding the orthogonal trajectories of the given family of curves in Cartesian coordinates (i.e. $f(x,y,c)=0$):

Step(i): Form the diff. equation $F\left(x,y,\frac{dy}{dx}\right)=0$ by eliminating 'c' from the given family of curves $f(x,y,c)=0$.

Step(2): Replace $\frac{dy}{dx}$ by $\frac{-1}{\frac{dy}{dx}}$ in $F(x,y,\frac{dy}{dx})=0$.

to get the diff equation $F(x,y,-\frac{dx}{dy})=0$ of the family of orthogonal trajectories.

Step(3): solve the diff. equation

$F(x,y,\frac{-dx}{dy})=0$ to get the family of orthogonal trajectories.

Problems:

→ Find the orthogonal trajectories of the family of curves $x^2+y^2=a^2$; $a$ is parameter.

Sol'n: The given family of curves is

$$x^2+y^2=a^2; \quad \text{—— (1)} \quad a \text{ is parameter.}$$

Diff. w.r.t '$x$', we get

$$2x+2yy'=0$$

$$\Rightarrow x+yy'=0 \quad \text{—— (2)}$$

which is the differential equation of the given family of curves (1).

Replacing "$y'$" by $-1/y'$ in (2), we get the diff. equation of the family of orthogonal trajectories

$$\therefore x-\frac{y}{y'}=0$$

$$\Rightarrow xy'=y$$

$$\Rightarrow \frac{1}{y}dy=\frac{1}{x}dx$$

$$\Rightarrow \log y=\log x+\log m$$

$$\Rightarrow \boxed{y=mx}; \quad m \text{ is parameter.}$$

which is the required orthogonal trajectories.

→ Find the orthogonal trajectories of the following family of curves.

(i) $y=ax^2$; $a$ is Parameter.

(ii) $3xy=x^3-a^3$; $a$ is parameter.

Now replacing $y'$ by $-\frac{1}{y'}$ in ③ we get the differential equation of orthogonal trajectories.

$$\therefore y = 2x\left(-\frac{1}{y'}\right) + y\left(-\frac{1}{y'}\right)^2$$

$$\Rightarrow y = \frac{-2x}{y'} + \frac{y}{y'^2}$$

$$\Rightarrow y(y')^2 = -2xy' + y$$

$$\Rightarrow y = 2xy' + y(y')^2 \quad \text{——} \quad ④$$

which is the same as the differential equation of the given family of parabolas. i.e., ③ & ④ are same.

$\therefore$ The system of parabolas ① is self orthogonal.

193 → show that the system of confocal conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is self-orthogonal; $\lambda$ is parameter.

(or)

2003 → Show that the orthogonal trajector of a system of confocal ellipses is self-orthogonal.

Sol$^n$: Given equation is

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \quad \text{——} \quad ①$$

Diff ① w.r.t '$x$' we get

$$\frac{2x}{a^2+\lambda} + \frac{2yy'}{b^2+\lambda} = 0 \Rightarrow x[b^2+\lambda] + yy'[a^2+\lambda] = 0$$

$$\Rightarrow b^2x + a^2yy' + \lambda(x+yy') = 0$$

$$\Rightarrow \lambda = \frac{-(b^2x+a^2yy')}{x+yy'}$$

Now $a^2+\lambda = a^2 - \frac{(b^2x+a^2yy')}{x+yy'}$

(iii) $x^2+y^2=2ax$; $a$ is parameter

(iv) $\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$; $\lambda$ is a parameter.

(v) $\frac{x^2}{a^2}+\frac{y^2}{a^2+\lambda}=1$; $\lambda$ is a parameter

(vi) $x^2+y^2+2gx+c=0$; $g$ is parameter

12 → $ay^2=x^3$; $a$ is parameter.

## Self-orthogonal family of Curves:

A family of curves is said to be self orthogonal if the diff. equation of the ~~orthogonal trajectories~~ family of curves is same as the diff. eqn of of the orthogonal trajectories.

(or)

If each member of a given family of curves ~~is said to be self orthogonal~~ intersects all other members orthogonally, then the given family of curves is said to be self-orthogonal.

**Problems:**

→ Show that the system of confocal and co-axial parabolas $y^2=4a(x+a)$ is self-orthogonal, a being parameter.

$y^2=4a(x+a)$ —— (1)

Differentiating (1) w.r.t '$x$' we get,

$$2yy'=4a \quad \text{—— (2)}$$

Now eliminating '$a$' b/w (1) & (2), we get

$$y^2=2yy'\left(x+\frac{yy'}{2}\right)$$

$$\Rightarrow y^2=2xyy'+y^2(y')^2$$

$$\Rightarrow y=2xy'+y(y')^2 \quad \text{—— (3)}$$

which is the differential equation of the given family of Parabolas.

$$= \frac{a^2x + a^2yy' - b^2x - a^2yy'}{x+yy'}$$

$$\Rightarrow \boxed{a^2+\lambda = \frac{(a^2-b^2)x}{x+yy'}}$$

and $$b^2+\lambda = b^2 - \frac{(b^2x + a^2yy')}{x+yy'}$$

$$= \frac{b^2x + b^2yy' - b^2x - a^2yy'}{x+yy'}$$

$$= \frac{-(a^2-b^2)yy'}{x+yy'}$$

$$① \equiv x^2\left[\frac{x+yy'}{(a^2-b^2)x}\right] - y^2\left[\frac{x+yy'}{(a^2-b^2)yy'}\right] = 1$$

$$xy'(x+yy') - y(x+yy') = (a^2-b^2)y'$$

$$\Rightarrow (x+yy')(xy'-y) = (a^2-b^2)y'$$

$$\Rightarrow (x+yy')\left(x - \frac{y}{y'}\right) = (a^2-b^2) \quad \text{——— ②}$$

which is the diff of the given system of conics ①.

Now replacing $y'$ by $\frac{-1}{y'}$ in ②

we get the differential equation of the family of orthogonally trajectories.

$$\left(x - \frac{y}{y'}\right)(x+yy') = a^2-b^2 \quad \text{——— ③}$$

② & ③ are same

$\therefore$ The given system of confocal conics is self orthogonal.

**Orthogonal trajectories in polar co-ordinates (i.e, $f(r,\theta,c)=0$):**

**Working Rule:**

Step(1): Form the diff equation $F\left(r,\theta,\frac{dr}{d\theta}\right)=0$ by eliminating 'c' from the given family of curves $f(r,\theta,c)=0$.

Step (2): Replace $\frac{dr}{d\theta}$ by $\frac{-r^2}{\frac{dr}{d\theta}}$ $\left((\text{or}) \; -r^2\frac{d\theta}{dr}\right)$ in

$F\left(r,\theta,\frac{dr}{d\theta}\right)=0$ to get the diff. equation $F\left(r,\theta,-r^2\frac{d\theta}{dr}\right)=0$

of the family of orthogonal trajectories.

Step(3): solve the diff. equation $F\left(r,\theta,r^2\frac{d\theta}{dr}\right)=0$ to get the orthogonal trajectories.

Problems:

→ Find the orthogonal trajectories of family of Cardioids $r=a(1-\cos\theta)$; where $a$ is the parameter.

Sol$^n$: Given family of Cardioids is $r=a(1-\cos\theta)$ —— ①

Taking log on bothsides we get

$$\log r = \log a + \log(1-\cos\theta) \quad \text{—— ②}$$

Diff. ② w.r.t $\theta$, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1-\cos\theta}(\sin\theta)$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} \quad \text{—— ③}$$

which is the diff equation of the given family ①.

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in ③, we get the differential equation of the orthogonal trajectories.

$$\therefore \frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{\sin\theta}{1-\cos\theta}$$

$$\Rightarrow -r\frac{d\theta}{dr} = \frac{2\sin\theta/2\,\cos\theta/2}{2\sin^2\theta/2}$$

$$\Rightarrow -r\frac{d\theta}{dr} = \cot(\theta/2)$$

$$\Rightarrow \frac{dr}{r} = -\tan(\theta/2)\,d\theta$$

$$\Rightarrow \log r = \frac{-\log(\sec(\theta/2))}{1/2} + c$$

$$\Rightarrow \log(r/c) = \log|\cos^2(\theta/2)|$$

$$\Rightarrow r/c = \cos^2(\theta/2)$$

$$\Rightarrow r/c = \left(\frac{1+\cos\theta}{2}\right)$$

$$\Rightarrow r = \frac{c}{2}(1+\cos\theta)$$

$$\Rightarrow r = b(1+\cos\theta) \quad (\text{Put } c/2 = b)$$

which is the required orthogonal trajectories of the following family of curves.

(1) $r = \dfrac{2a}{1+\cos\theta}$ ; $a$ is parameter

(2) $r^n \cos n\theta = a^n$ ; $a$ is parameter.

**Working rule for finding the oblique trajectories of the given family of curves in Cartisian co-ordinates (i.e. $f(x,y,c)=0$):**

Step(1): Form the diff. equation $F(x,y,\frac{dy}{dx})=0$ by eliminating $c$ from the given family of curves. $f(x,y,c)=0$.

Step(2): Replace $\frac{dy}{dx}$ (i.e. $p$) by $\dfrac{p+\tan\alpha}{1-p\tan\alpha}$ (or) $\dfrac{1+p\cot\alpha}{\cot\alpha - p}$ in $F(x,y,\frac{dy}{dx})=0$ to get the diff. equation $F\left(x,y,\dfrac{p+\tan\alpha}{1-p\tan\alpha}\right)=0$ where $p=\frac{dy}{dx}$ of the family of trajectories.

Step(3): Solve the diff. equation $F\left(x,y\,\dfrac{p+\tan\alpha}{1-p\tan\alpha}\right)=0$ to get the family trajectories.

Problems:

→ Find the $45^\circ$ trajectories of the family of curves $xy=c$. where $c$ is parameter.

Sol'n: Given family of curves is $xy = c$ —— ①

Diff. w.r.t 'x', we get

$$xy' + y = 0 \quad \text{——} ②$$

$$(\text{or}) \quad xp + y = 0$$

Replacing 'p' by $\dfrac{p+\tan\alpha}{1-p\tan\alpha}$ in ②.

we get,

$$\frac{x(p+\tan\alpha)}{1-p\tan\alpha} + y = 0$$

$$\Rightarrow x\left(\frac{p+\tan 45^\circ}{1-p\tan 45^\circ}\right) + y = 0$$

$$\Rightarrow x\left(\frac{p+1}{1-p}\right) + y = 0$$

$$\Rightarrow x(p+1) + y(1-p) = 0$$

$$\Rightarrow (x-y)p + (x+y) = 0$$

$$\Rightarrow p = \frac{y+x}{y-x}$$

$$\Rightarrow (y+x)dx + (x-y)dy = 0 \quad \text{——} ③$$

clearly which is the form of $Mdx + Ndy = 0$.

where $M = y + x$ ; $N = x - y$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 1$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Integrating ③ we get

$$yx + \frac{x^2}{2} - \frac{y^2}{2} = c$$

which is the required trajectory of the given family.

→ Determine the $45^\circ$ trajectories of the family of concentric circles $x^2+y^2=a^2$; a is the parameter

$$\Rightarrow (xp-y)^2 = (x^2-y^2)\sin^{-1}(y/x) \quad \text{——(1)}$$

putting $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\Rightarrow p = v + xP$ where $P = \frac{dv}{dx}$.

$\therefore$ (1) $\Rightarrow$

$$[x(v+xP) - y]^2 = (x^2 - x^2v^2)\sin^{-1}v$$

$$\Rightarrow (y + x^2P - y)^2 = x^2(1-v^2)\sin^{-1}v$$

$$\Rightarrow (x^2P)^2 = x^2(1-v^2)\sin^{-1}v$$

$$\Rightarrow x^2P^2 = (1-v^2)\sin^{-1}v$$

$$\Rightarrow P^2 = \frac{1-v^2}{x^2}\sin^{-1}v$$

$$\Rightarrow P = \pm\frac{\sqrt{1-v^2}}{x}\sqrt{\sin^{-1}v}$$

$$\Rightarrow \frac{dv}{dx} = \pm\frac{\sqrt{1-v^2}}{x}\sqrt{\sin^{-1}v}$$

$$\Rightarrow \int\frac{1}{\sqrt{1-v^2}\sqrt{\sin^{-1}v}}\,dv = \pm\int\frac{dx}{x} + C$$

$\sin^{-1}v = t$

$$\Rightarrow \frac{1}{\sqrt{1-v^2}}dv = dt$$

$$\int\frac{dt}{\sqrt{t}} = \pm\log x + C$$

$$\Rightarrow 2t^{1/2} = \pm\log x + C$$

$$\Rightarrow 4t = (\pm\log x + C)^2$$

$$\Rightarrow 4\sin^{-1}v = (\pm\log x + C)^2$$

$$\Rightarrow 4\sin^{-1}(y/x) = (\pm\log x + C)^2$$

which is the required solution

$$\rightarrow xp^2 - (y-x)p - y = 1 \quad \text{——(1)}$$

$$\Rightarrow px(p+1) - y(p+1) = 1$$

$$\Rightarrow (px-y)(p+1) = 1$$

$$\Rightarrow px - y = \frac{1}{p+1}$$

$$\Rightarrow y = px - \frac{1}{p+1} \quad \text{——(2)}$$

Clearly which is in the form of Clairaut's equation.

put $p = c$, we get

$$\boxed{y = cx - \frac{1}{c+1}} \quad \text{——(3)}$$

which is the G.S. of (1)

Due to the Clairaut's form of (2), the p-discriminant and c-discriminant relations are same.

$\therefore$ from (1)

p-discriminant is given by

$$(y-x)^2 + 4xy = 0$$

$$\Rightarrow (x+y)^2 = 0$$

which must be the singular solution because it is present in both the discriminants and satisfies the given differential equation.

$\Rightarrow (xp-y)^2 = a(1+p^2)(x^2+y^2)^{3/2}$ —— (1)

putting $x = r\cos\theta$ ; $y = r\sin\theta$

$\therefore x^2+y^2 = r^2$

$\Rightarrow x\,dx + y\,dy = r\,dr$

$\Rightarrow x + yp = r\dfrac{dr}{dx}$

and $\dfrac{y}{x} = \tan\theta$

$\Rightarrow \dfrac{x\,dy - y\,dx}{x^2} = \sec^2\theta\, d\theta$

$\Rightarrow xp - y = r^2\dfrac{d\theta}{dx}$

Now $\dfrac{x+yp}{xp-y} = \dfrac{1}{r}\dfrac{dr}{d\theta}$

$\Rightarrow \dfrac{x+yp}{xp-y} = \dfrac{1}{r}P$ where $\dfrac{dr}{d\theta} = P$

$\Rightarrow rx + ryp = Ppx - Py$

$\Rightarrow (\cancel{Px-ry})p = rx + Py$

$\Rightarrow p = \dfrac{rx+Py}{Px-ry}$

$\therefore$ (1)

$$\left[x\left(\frac{rx+Py}{Px-ry}\right) - y\right]^2 = a\left[1+\left(\frac{rx+Py}{Px-ry}\right)^2\right](r^2)^{3/2}$$

$\Rightarrow (x^2r + \cancel{xPy} - \cancel{Pxy} + ry^2)^2 = a\dfrac{[(Px-ry)^2+(rx+Py)^2]}{(r^2)}$

$\Rightarrow (x^2+y^2)^2 r^2 = a[(P^2+r^2)(x^2+y^2)]r^3$

$\Rightarrow (x^2+y^2) = a(P^2+r^2)r$

$\Rightarrow r^2 = a(P^2+r^2)r$

$\Rightarrow r = a(P^2+r^2)$

$\Rightarrow aP^2 = r - ar^2$

$\Rightarrow P^2 = \dfrac{r-ar^2}{a}$

$\Rightarrow P = \pm\dfrac{\sqrt{r(1-ar)}}{\sqrt{a}}$

$\Rightarrow \dfrac{dr}{d\theta} = \pm\dfrac{\sqrt{r}\sqrt{1-ar}}{\sqrt{a}}$

$\Rightarrow \displaystyle\int \frac{\sqrt{a}}{\sqrt{r}\sqrt{1-ar}}\,dr = \pm\int d\theta$

put $1-ar = t$ and $r = \dfrac{1-t}{a}$

$\Rightarrow -a\,dr = dt$

$\Rightarrow dr = -\dfrac{1}{a}dt$

$\displaystyle\int \frac{\sqrt{a}\left(-\frac{1}{a}dt\right)}{\sqrt{\frac{1-t}{a}}\;\sqrt{t}} = \pm\theta + C$

$-\displaystyle\int \frac{dt}{\sqrt{t(1-t)}} = \pm\theta + C$

$-\displaystyle\int \frac{dt}{\sqrt{t-t^2}} = \pm\theta + C$

$-\displaystyle\int \frac{dt}{\sqrt{\left(t-\frac{1}{2}\right)^2 + \frac{1}{4}}} = \pm\theta + C$

$\Rightarrow \sin^{-1}\left(\dfrac{t-\frac{1}{2}}{\frac{1}{2}}\right) = \pm\theta + C$

$\Rightarrow \sin^{-1}(2t-1) = \pm\theta + C$

$\Rightarrow \sin^{-1}[2(1-ar)-1] = \pm\theta + C$

$\Rightarrow 1 - 2ar = \sin^{-1}(\pm\theta + C)$

$\Rightarrow 2a\sqrt{x^2+y^2} - 1 = \sin^{-1}\left(\pm\tan^{-1}\frac{y}{x} + C\right)$

$\therefore 2a\sqrt{x^2+y^2} - 1 = \sin^{-1}\left(\pm\tan^{-1}\frac{y}{x} + C\right)$

which is the required solution

Set - V



# * The Laplace Transform *

## Introduction:

Laplace transform ~~or laplace~~ transformation is a method for solving linear differential equations and satisfying given boundary conditions without use of a general solution.

These particular solutions are the ones widely used in physics, mechanics, chemistry, medicine, national defence and many fields of pratical research. The knowledge of laplace transforms in recent years has an essential part of mathematical background required for engineers and scientists.

## * Integral Transform:

Let $K(P,t)$ be a function of two variables $P$ and $t$, where $P$ is a parameter (may be real (or) complex) independent of $t$. The function $f(P)$ defined by the integral (assume to be convergent)

$f(P) = \int_{-\infty}^{\infty} K(P,t)F(t)\,dt$, is called the integral transform of the function $F(t)$ and is denoted by $T\{F(t)\}$.

The function $K(P,t)$ is called the kernel of the transformation.

## * Laplace Transform:

If the kernal $K(P,t)$ is defined as

$$K(P,t) = \begin{cases} 0 & \text{for } t<0 \\ e^{-Pt} & \text{for } t \geq 0 \end{cases}$$

then $f(P) = \int_{0}^{\infty} e^{-Pt}F(t)\,dt$ ——— (1)

$$f(P) = \int_{-\infty}^{\infty} K(P,t)F(t)\,dt = \int_{-\infty}^{0} (0)\,F(t)\,dt + \int_{0}^{\infty} e^{-Pt}F(t)\,dt = \int_{0}^{\infty} e^{-Pt}F(t)\,dt$$

The function $f(P)$ defined by the integral (1) is called the Laplace transform of the function $F(t)$ and is denoted by $L\{F(t)\}$ (or) $\bar{F}(P)$ (or) $L[F(t)]$

i.e., $L[F(t)] = \int_{0}^{\infty} e^{-Pt}F(t)\,dt$

Thus Laplace transform is a

function of a new variable (or parameter) 'P' given by (1).

Note: The Laplace transform of $F(t)$ is said to exist if the integral (1) converges for some values of P, otherwise it does not exist.

## * Linearity Property of Laplace Transformation:

A transformation T is said to be linear if for every pair of functions $F_1(t)$ and $F_2(t)$ and for every pair of constants $a_1$ and $a_2$.

we have

$$T\{a_1F_1(t)+a_2F_2(t)\}=a_1T\{F_1(t)\}+a_2T\{F_2(t)\}$$

→ The Laplace transformation is a linear transformation.

i.e, $L\{a_1F_1(t)+a_2F_2(t)\}$

$$= a_1L\{F_1(t)\}+a_2L\{F_2(t)\}$$

where $a_1$ & $a_2$ are constants.

Sol'n: We have

$$L\{F(t)\} = \int_0^\infty e^{-Pt}F(t)\,dt$$

$\therefore L\{a_1F_1(t)+a_2F_2(t)\}$

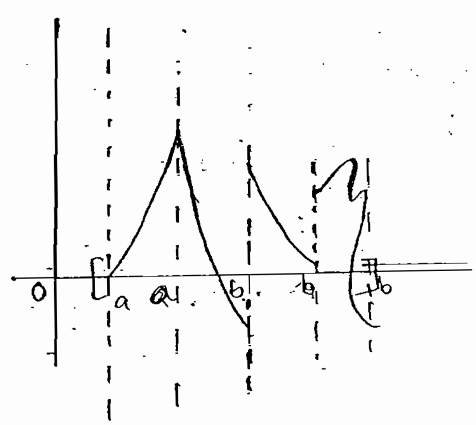
$$= \int_0^\infty e^{-Pt}\{a_1F_1(t)+a_2F_2(t)\}dt$$

$$= a_1\int_0^\infty e^{-Pt}F_1(t)dt + a_2\int_0^\infty e^{-Pt}F_2(t)dt$$

$$= a_1L\{F_1(t)\}+a_2L\{F_2(t)\}$$

## → Piecewise (or) Sectionally Continuous Function:

A function $F(t)$ is said to be piecewise (or sectionally) continuous on $t\in[a,b]$, if it defined on that interval and is such that the interval can be subdivided into finite number of intervals, in each of which $F(t)$ is continuous and has finite right and left hand limits.



## Existence of Laplace Transform:

If $F(t)$ is a function which is piecewise continuous on every finite interval in the range $t \geq 0$ and satisfies

$|F(t)| \leq Me^{at}$ for all $t \geq 0$ and forsome constants $a$ and $M$, then the laplace transform of $F(t)$ exists for all $p > a$.

**Proof:** we have

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^{t_0} e^{-pt} F(t)\,dt + \int_{t_0}^\infty e^{-pt} F(t)\,dt \quad \text{—— (1)}$$

The integral $\int_0^{t_0} e^{-pt} F(t)\,dt$ exists since $F(t)$ is piecewise continuous on every finite interval $0 \leq t \leq t_0$.

Now $\left|\int_{t_0}^\infty e^{-pt} F(t)\,dt\right| \leq \int_{t_0}^\infty |e^{-pt} F(t)|\,dt$

$$\leq \int_{t_0}^\infty e^{-pt} Me^{at}\,dt$$

since $|F(t)| \leq Me^{at}$

$$= \int_{t_0}^\infty e^{-(p-a)t} M\,dt$$

$$= \left. \frac{-e^{-(p-a)t} M}{(p-a)} \right|_{t_0}^\infty$$

$$= \frac{Me^{-(p-a)t_0}}{p-a}; \quad p > a$$

$$\therefore \left|\int_{t_0}^\infty e^{-pt} F(t)\,dt\right| \leq \frac{Me^{-(p-a)t_0}}{p-a}; \quad p > a$$

But $\frac{Me^{-(p-a)t_0}}{p-a}$ can be made as small as we please by taking $t_0$ sufficiently large.

Thus from (1), we conclude that $L\{F(t)\}$ exists for all $p > a$.

**Note (1):** Above theorem of existence of laplace transform can also be stated as:

"If $F(t)$ is a function which is piece-wise continuous on every finite interval in the range $t \geq 0$ and is of exponential order '$a$' as $t \to \infty$, the laplace transform of $F(t)$ exists for all $p > a$".

(or)

"If $F(t)$ is a function of class A, the laplace transform of $F(t)$ exists

for all $p > a$".

Note(2): Conditions in the above theorem are sufficient but not necessary for the existence of the Laplace transform. If these conditions are satisfied, the Laplace transform must exist.

If these conditions are not satisfied, the Laplace transform may or may not exist.

For eg: Consider the function

$$F(t) = \frac{1}{\sqrt{t}}$$

Here $F(t) \to \infty$ as $t \to 0$ from the right. Thus the function $F(t)$ is not piece wise continuous on every finite interval in the range $t \geq 0$.

But $F(t)$ is integrable from $0$ to any positive value $t_0$.

Also $|F(t)| < Me^{at}$ for all $t > 1$ with $M = 1$ and $a = 0$.

Now $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\, dt$

$$= \int_0^\infty e^{-pt} \frac{1}{\sqrt{t}}\, dt \quad \text{which is converges for } p>0.$$

$$= \frac{2}{\sqrt{p}} \int_0^\infty e^{-x^2} dx$$

Putting $\sqrt{pt} = x \Rightarrow pt = x^2$

$\sqrt{p}\, \frac{1}{2\sqrt{t}} dt = dx$

$\Rightarrow \frac{dt}{\sqrt{t}} = \frac{2}{\sqrt{p}} dx$

Limits of $x$: if $t = 0 \Rightarrow x = 0$, $t = \infty \Rightarrow x = \infty$

$$= \frac{2}{\sqrt{p}} \cdot \frac{\sqrt{\pi}}{2} \qquad \left(\because \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}\right)$$

$$= \sqrt{\frac{\pi}{p}}, \; p > 0$$

$\therefore L\left\{\frac{1}{\sqrt{t}}\right\}$ exists for $p > 0$ even if $F(t) = \frac{1}{\sqrt{t}}$ is not piece-wise continuous in the range $t \geq 0$.

## * Functions of Exponential order:

A function $F(t)$ is said to be of exponential order $\alpha$ as $t \to \infty$ if there exists a +ve constant (real) $M$, a number $\alpha$ and a finite number $t_0$ such that

$$|F(t)| < Me^{\alpha t} \quad \forall \; t \geq t_0.$$

(or) $|e^{-\alpha t} F(t)| < M \quad \forall \; t \geq t_0$.

If a function $F(t)$ is of exponential order $\alpha$, it also of $\beta$, $\beta > \alpha$ (OR)

A function $F(t)$ is said to be exponential order $\alpha$ as $t \to \infty$ if $\lim_{t \to \infty} e^{-\alpha t} F(t) = $ finite quantity.

## * Function of Class A:

A function $F(t)$ is said to be function of class A if (i) it is piecewise (or sectionally) continuous on every finite interval in the range $t \geq 0$.

(ii) $F(t)$ is of exponential order as $t \to \infty$

→ Prove that $F(t) = t^n$ is of exponential order as $t \to \infty$, 'n' being any +ve integer.

Sol'n: $\underset{t\to\infty}{Lt} \{e^{-at} F(t)\} = \underset{t\to\infty}{Lt}\ e^{-at} t^n,\ a>0$

$= \underset{t\to\infty}{Lt} \frac{t^n}{e^{at}}$, $\left(\frac{\infty}{\infty}\text{ form}\right)$

$= \underset{t\to\infty}{Lt} \frac{n!}{a^n e^{at}}$ (by L.Hospital rule)

$= \frac{n!}{\infty} = 0$

$\therefore \underset{t\to\infty}{Lt}\ e^{-at} t^n = 0 =$ finite number

$\therefore t^n$ is of exponential order as $t \to \infty$

Note: $|F(t)| = t^n < e^{nt}\ \forall\ t>0$.

$\therefore$ The given function is of exponential order n.

Ex: show that $t^2$ is of exponential order 3.

Sol'n: we have

$\underset{t\to\infty}{Lt} \{e^{-at} F(t)\} = \underset{t\to\infty}{Lt}\left(\frac{t^2}{e^{at}}\right)$

$= \underset{t\to\infty}{Lt} \frac{2t}{ae^{at}}$ (By L-Hospital rule)

$= \underset{t\to\infty}{Lt} \frac{2}{a^2 e^{at}}$ (By L-Hospital rule)

$= 0$, if $a>0$

$\therefore F(t) = t^2$ is of exponential order.

Now $|t^2| = t^2 < e^{2t} < e^{3t}\ \forall\ t>0$

$\therefore$ The given function is of exponential order 3. ($\because$ if $F(t)$ is of exponential order $\alpha = 2$ it is also of $\beta = 3 \because 3>2$)

→ Show that the function $e^{t^2}$ is not of exponential order as $t\to\infty$

Sol'n: we have

$\underset{t\to\infty}{Lt} \{e^{-at} \cdot F(t)\} = \underset{t\to\infty}{Lt} \{e^{-at} e^{t^2}\}$

$= \underset{t\to\infty}{Lt}\ e^{t(t-a)}$

$= \infty\ \forall a$.

Hence whatever be the value of a, we cannot find a number M such that $e^{t^2} < Me^{at}$.

$\therefore$ The given function is not of exponential order as $t\to\infty$.

→ Find the Laplace transform of the function $F(t) = 1$

Sol'n: we have $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\, dt$

$$\therefore L\{1\} = \int_0^\infty e^{-pt} \cdot 1\, dt$$

$$= \left[\frac{-e^{-pt}}{p}\right]_0^\infty \quad (\because e^{-pt} \to 0 \text{ as } t \to \infty)$$

$$= \frac{1}{p}, \; p > 0$$

Note: Here the condition $p>0$ is necessary. Since the integral is convergent for $p>0$ and divergent for $p \le 0$.

→ Find $L\{t^n\}$, $n$ is +ve integer.

Sol'n: we have $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\, dt$

$$\therefore L\{t^n\} = \int_0^\infty e^{-pt}\, t^n\, dt$$

Integrating by parts

$$= \left[-\frac{1}{p} t^n e^{-pt}\right]_0^\infty + \frac{1}{p}\int_0^\infty n t^{n-1} e^{-pt} dt$$

$$= -\frac{1}{p} \lim_{t\to\infty} \frac{t^n}{e^{pt}} + 0 + \frac{n}{p}\int_0^\infty e^{-pt} t^{n-1} dt$$

$$= 0 + \frac{n}{p}\int_0^\infty e^{-pt} t^{n-1} dt$$

$(\because \lim_{t\to\infty} \frac{t^n}{e^{pt}} = 0$ by L-Hospital's rule$)$

$$= \frac{n}{p}\int_0^\infty e^{-pt}\, t^{n-1}\, dt$$

Proceeding Similarly, we get

$$L\{t^n\} = \frac{n!}{p^n}\int_0^\infty e^{-pt} dt$$

$$= \frac{n!}{p^n}\left[\frac{-e^{-pt}}{p}\right]_0^\infty$$

$$= -\frac{n!}{p^n}\left[0 - \frac{1}{p}\right]$$

$$= \frac{n!}{p^{n+1}}, \; p > 0.$$

→ Show that the Laplace transform of the function $F(t) = t^n$, $-1 < n < 0$, exists, although it is not a function of class A.

Sol'n: Given $F(t) = t^n$, $-1<n<0$

$n = -\frac{1}{2}$, $F(t) = \frac{1}{\sqrt{t}}$

Here $F(t) \to \infty$ as $t \to 0$ (for $t \ge 0$)

i.e., the function is not piecewise continuous on every finite interval in the range $t \ge 0$.

we have $\lim_{t\to\infty}\{e^{-at} F(t)\} = \lim_{t\to\infty}\left(\frac{t^n}{e^{at}}\right)$

$$= \lim_{t\to\infty} \frac{1}{t^{-n} e^{at}}$$

$$= \lim_{t\to\infty} \frac{1}{t^m e^{at}},$$

where $0<m<1$

$= 0$, if $a>0$.

$-1<n<0$

$1>-n>0$

$1>m>0$

Put $m=-n$

$\therefore F(t) = t^n$ is of exponential order

Since $F(t) = t^n$ is not sectionally continuous over every finite interval in the range $t \geq 0$.

$\therefore$ It is not a function of class A.

But $t^n$ is integrable from 0 to any +ve number $t_0$.

Now $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\, dt$

$= \int_0^\infty e^{-pt} t^n\, dt$

$= \int_0^\infty e^{-x} \left(\frac{x}{p}\right)^n \frac{1}{p} dx;$ Putting $pt = x$, $p\,dt = dx$ $\Rightarrow dt = \frac{dx}{p}$ & taking B.O.

$= \frac{1}{p} \cdot \frac{1}{p^n} \int_0^\infty e^{-x} x^n\, dx$

$= \frac{1}{p^{n+1}} \int_0^\infty e^{-x} x^{(n+1)-1} dx$

(By definition of Gamma function

$= \frac{\Gamma(n+1)}{p^{n+1}}$, $\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ if $n>0$)

if $p>0$ and $n+1>0$

i.e., $n>-1$.

Hence Laplace transform of $t^n$, $0 > n > -1$ exists, although it is not a function of class A.

$\rightarrow$ Find $L\{e^{at}\}$.

Sol'n: Here $L\{e^{at}\} = \int_0^\infty e^{-pt} e^{at} dt$

$= \int_0^\infty e^{-(p-a)t} dt$

$= \left[\frac{-e^{-(p-a)t}}{p-a}\right]_0^\infty, \; p>a$

$= \frac{1}{p-a}$

$\rightarrow$ Find $L\{\cos at\}$ and hence obtain $L(\sin^2 at)$

Sol'n: $L\{\cos at\} = \int_0^\infty e^{-pt} \cos at\, dt$

$$\boxed{\begin{array}{l}\int e^{ax} \sin bx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \\ \int e^{ax}\cos bx\, dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}\end{array}}$$

$= \left[\frac{e^{-pt}}{p^2+a^2}(-p\cos at + a\sin at)\right]_0^\infty$

$= 0 - \left(\frac{-p}{p^2+a^2}\right), \; p>0$ ($\because$ as $t\to\infty$, $e^{-pt}\to 0$)

$= \frac{p}{p^2+a^2}, \; p>0$ —— (1)

Now $L(\sin^2 at) = L\left\{\frac{1-\cos 2at}{2}\right\}$

$= \frac{1}{2} L\{1 - \cos 2at\}$

$= \frac{1}{2} L\{1\} - \frac{1}{2} L\{\cos(2a)t\}$

$= \frac{1}{2p} - \frac{1}{2} \frac{p}{p^2+(2a)^2}$ ($\because$ by (1))

$$= \frac{1}{2p} - \frac{p}{2(p^2+4a^2)}$$

$$= \frac{2a^2}{p(p^2+4a^2)}$$

→ Find $L\{\cosh at\}$

Sol'n: $L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$$= \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\}$$

$$= \frac{1}{2}\cdot\frac{1}{p-a} + \frac{1}{2}\cdot\frac{1}{p+a}$$

$$= \frac{p}{p^2-a^2}, \ p>a \ \& \ p>-a$$

i.e. $p>|a|$

i.e. $|a|<p$.

→ Find (1) $L\{\sin at\}$ (2) $L\{\sinh at\}$

→ Find $L\{\sin t\cos t\}$.

Sol'n: Given $L\{\sin t \cos t\} = L\{\frac{1}{2}\sin 2t\}$

$$= \frac{1}{2}L\{\sin 2t\}$$

$$= \frac{1}{2}\cdot\frac{2}{p^2+4}, \ p>0$$

$$= \frac{1}{p^2+4}, \ p>0.$$

→ Find (1) $L\{\cosh^2 2t\}$

(2) $L\{7e^{2t}+9e^{-2t}+5\cos t+7t^3+5\sin 3t+2\}$

→ Find $L\{F(t)\}$, where $F(t)=\begin{cases}0, & 0<t<1\\ t, & 1<t<2\\ 0, & t>2\end{cases}$

Sol'n: Here $F(t)$ not defined at $t=0$, $t=1$ & $t=2$.

$$\therefore L\{F(t)\} = \int_0^\infty e^{-pt}F(t)\,dt$$

$$= \int_0^1 e^{-pt}\cdot 0\,dt + \int_1^2 e^{-pt}\,t\,dt + \int_2^\infty e^{-pt}\cdot 0\,dt$$

$$= \int_1^2 e^{-pt}\,t\,dt$$

$$= \left[-t\frac{e^{-pt}}{p}\right]_1^2 - \int_1^2 \frac{e^{-pt}}{-p}\,dt$$

$$= -\frac{2}{p}e^{-2p} + \frac{e^{-p}}{p} - \left[\frac{e^{-pt}}{p^2}\right]_1^2$$

$$= -\frac{2}{p}e^{-2p} + \frac{e^{-p}}{p} - \frac{e^{-2p}}{p^2} + \frac{e^{-p}}{p^2}$$

$$= \left(\frac{1}{p}+\frac{1}{p^2}\right)e^{-p} - \left(\frac{1}{p^2}+\frac{2}{p}\right)e^{-2p}$$

→ Find the L.T. of the function $F(t)$, where $F(t) = \begin{cases}4, & 0<t<1\\ 3, & t>1\end{cases}$

→ Find the L.T of the function $F(t)$, where $F(t) = \begin{cases}2t & 0\le t\le 5\\ 1 & t>5\end{cases}$

→ Find the L.T of the function $F(t)$, where $F(t) = \begin{cases}\sin t, & 0<t<\pi\\ 0, & t>\pi\end{cases}$

→ Find $L\{\sin\sqrt{t}\}$

Sol'n: we have

$$L\{\sin\sqrt{t}\} = L\left\{\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \cdots\right\}$$

$$= L\left\{t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} - \cdots\right\}$$

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

$= L\{t^{1/2}\} - \frac{1}{3!}L\{t^{3/2}\} + \frac{1}{5!}L\{t^{5/2}\} - \frac{1}{7!}L\{t^{7/2}\} + \cdots$

$= \frac{\Gamma 3/2}{p^{3/2}} - \frac{1}{3!}\frac{\Gamma 5/2}{p^{5/2}} + \frac{1}{5!}\frac{\Gamma 7/2}{p^{7/2}} - \frac{1}{7!}\frac{\Gamma 9/2}{p^{9/2}} + \cdots$

$\left[\because L\{t^n\} = \frac{\Gamma n+1}{p^{n+1}} \text{ if } n > -1\right]$

$= \frac{\frac{1}{2}\Gamma\frac{1}{2}}{p^{3/2}} - \frac{1}{6}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{p^{5/2}} + \frac{1}{120}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{p^{7/2}} - \cdots$

$\left[\because \Gamma n+1 = n\Gamma n \right.$
$= n\Gamma(n-1)+1$
$= n(n-1)\Gamma(n-1)$
$\left.= n(n-1)(n-2)\Gamma(n-2) \text{ etc}\right]$

$= \frac{\frac{1}{2}\sqrt{\pi}}{p^{3/2}} - \frac{1}{6}\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{p^{5/2}} + \frac{1}{\cancel{120}_{\,48}}\frac{\frac{5}{2}\cdot\frac{\cancel{3}}{2}\cdot\frac{1}{2}\sqrt{\pi}}{p^{7/2}} + \cdots$

$(\because \Gamma\tfrac{1}{2} = \sqrt{\pi})$

$= \frac{\sqrt{\pi}}{2p^{3/2}}\left[1 - \frac{1}{4p} + \frac{1}{2!}\left(\frac{1}{4p}\right)^2 - \frac{1}{3!}\left(\frac{1}{4p}\right)^3 + \cdots\right]$

$= \frac{\sqrt{\pi}}{2p^{3/2}} e^{-1/4p}$

→ Find the L.T of the function $F(t) = (\sin t - \cos t)^2$.

→ Find the L.T of the function $F(t) = \frac{e^{at}-1}{a}$.

→ Evaluate $L\{F(t)\}$, if $F(t) = \begin{cases}(t-1)^2, & t>1\\ 0, & 0<t<1\end{cases}$

→ Find $L\{F(t)\}$, if $F(t) = \begin{cases} e^t, & 0<t\le 1\\ 0, & t>1\end{cases}$

→ Prove that $L\left[\frac{1}{\sqrt{\pi t}}\right] = \frac{1}{\sqrt{p}}$.

## * Laplace Transforms of some elementary Functions:

(1) $L\{1\} = \frac{1}{p}$, $p>0$

(2) $L\{t^n\} = \frac{n!}{p^{n+1}}$, $p>0$, where $n$ is +ve integer.

(3) $L\{t^n\} = \frac{\Gamma n+1}{p^{n+1}}$, $p>0$, where $n>-1$

(4) $L\{e^{at}\} = \frac{1}{p-a}$, $p>a$

(5) $L\{\sin at\} = \frac{a}{p^2+a^2}$, $p>0$

(6) $L\{\cos at\} = \frac{p}{p^2+a^2}$, $p>0$

(7) $L\{\sinh at\} = \frac{a}{p^2-a^2}$, $p>|a|$ i.e. $|a|<p$

(8) $L\{\cosh at\} = \frac{p}{p^2-a^2}$, $p>|a|$ i.e. $|a|<p$

Note: If $n$ is +ve integer, then $\Gamma n+1 = n!$

## * First Translation (or) shifting Theorem:

If $L\{F(t)\} = f(p)$, then

$L(e^{at}F(t)) = f(p-a)$.

**Proof:** By definition of Laplace Transform, we have

$L\{F(t)\} = f(P)$

$= \int_0^\infty e^{-Pt} F(t)dt$

$\therefore f(P-a) = \int_0^\infty e^{-(P-a)t} F(t)dt$

$= \int_0^\infty e^{-Pt}(e^{at} F(t))dt$

$= L\{e^{at} F(t)\}$

*** Second translation (or) shifting theorem:**

If $L\{F(t)\} = f(P)$,

and $G(t) = \begin{cases} F(t-a) & , t>a \\ 0 & , t<a \end{cases}$ then

$L\{G(t)\} = e^{-aP} f(P)$.

**Proof:** By definition of Laplace transformation, we have

$L(G(t)) = \int_0^\infty e^{-Pt} G(t)dt$

$= \int_0^a e^{-Pt} G(t)dt + \int_a^\infty e^{-Pt} G(t)dt$

$= \int_0^a e^{-Pt} \cdot 0\,dt + \int_a^\infty e^{-Pt} F(t-a)dt$

$= \int_a^\infty e^{-Pt} F(t-a)dt$ Putting $t-a=x$ $\Rightarrow t = a+x$ $\Rightarrow dt = dx$

$= \int_0^\infty e^{-P(a+x)} F(x)\,dx$

$= e^{-Pa}\int_0^\infty e^{-Px} F(x)dx$ $\left(\because \int_0^\infty e^{-Px} F(x)dx = \int_0^\infty e^{-Pt} F(t)dt\right.$

$= e^{-Pa}\int_0^\infty e^{-Pt} F(t)\,dt$ By property of definite integrals, i.e. $\left.\int_a^b f(x)dx = \int_a^b f(t)dt\right)$

$= e^{-Pa} L\{F(t)\}$

$= e^{-Pa} f(P)$

*** Change of scale Property**

**Theorem:** If $L\{F(t)\} = f(P)$, then

$L\{F(at)\} = \frac{1}{a} f(P/a)$.

**Proof:** By definition, we have

$L\{F(at)\} = \int_0^\infty e^{-Pt} F(at)dt$ Putting $at = x$ $\Rightarrow t = x/a$ $\Rightarrow dt = \frac{1}{a}dx$

$= \int_0^\infty e^{-P(x/a)} f(x)\frac{dx}{a}$

$= \frac{1}{a}\int_0^\infty e^{-(P/a)x} F(x)dx$

$= \frac{1}{a}\int_0^\infty e^{-(P/a)t} F(t)dt$ $\left(\because \int_a^b f(x)dx = \int_a^b f(t)dt\right)$

$= \frac{1}{a} f(P/a)$.

Find $L\{t^3 e^{-3t}\}$ i.e. $L\{\underbrace{e^{-3t}}_{e^{at}}\underbrace{t^3}_{F(t)}\}$

clearly which is in the form $L\{e^{at} F(t)\}$

**Sol'n:** Now $L\{F(t)\} = L\{t^3\}$

$= \frac{3!}{P^4} = \frac{6}{P^4} = f(P)$ (say)

$\therefore$ from first shifting theorem

$L\{e^{at} F(t)\} = f(P-a)$

$L\{t^3 e^{-3t}\} = \frac{6}{(P+3)^4}$ $\boxed{a=-3}$

→ Find $L\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$

Sol'n: we have

$L\{3\cos 6t - 5\sin 6t\} = 3L\{\cos 6t\} - 5L\{\sin 6t\}$

$= 3 \cdot \frac{P}{P^2+36} - 5 \cdot \frac{6}{P^2+36}$

$= \frac{3P-30}{P^2+36} = f(P)$ say

$\therefore$ from first shifting theorem, we have

$L\{e^{-2t}(3\cos 6t - 5\sin 6t)\} = f(P+2)$

$= \frac{3(P+2)-30}{(P+2)^2+36}$ $\boxed{a=-2}$

If $L\{F(t)\} = f(P)$ then $L\{e^{at}F(t)\} = f(P-a)$

$= \frac{3P-24}{P^2+4P+40}$

(OR)

By definition of L.T.

$L\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$

$= \int_0^\infty e^{-Pt} e^{-2t}(3\cos 6t - 5\sin 6t)dt$

$= 3\int_0^\infty e^{-(P+2)t}\cos 6t - 5\int_0^\infty e^{-(P+2)t}\sin 6t \, dt.$

$= \left\{3 \cdot \frac{e^{-(P+2)t}}{(P+2)^2+6^2}\left[-(P+2)\cos 6t + 6\sin 6t\right]\right\}_0^\infty$

$- \left[5 \frac{e^{-(P+2)t}}{(P+2)^2+6^2}\left\{-(P+2)\sin 6t - 6\cos 6t\right\}\right]_0^\infty$

$= 0 + \frac{3(P+2)}{(P+2)^2+6^2} - 5\left[0 - \frac{(-6)}{(P+2)^2+6^2}\right]$

$= \frac{3(P+2)}{(P+2)^2+36} - \frac{30}{(P+2)^2+36} = \frac{3P-24}{P^2+4P+40}$

→ Find $L\{e^t(3\sinh 2t - 5\cosh 2t)\}$

→ Find (i) $L\{e^t \sin^2 t\}$ (ii) $L(e^t(t+3)^2)$

→ Find $L\left\{\frac{e^{-at}t^{n-1}}{(n-1)!}\right\}$, where n is +ve integer.

Sol'n: we have

$L\left\{\frac{t^{n-1}}{(n-1)!}\right\} = \frac{(n-1)!}{p^n(n-1)!}$ $\left(\because L(t^n) = \frac{n!}{p^{n+1}}\right)$

$= \frac{1}{p^n} = f(P)$, say

$\therefore L\left\{e^{-at}\frac{t^{n-1}}{(n-1)!}\right\} = \frac{1}{(P+a)^n}$

→ Applying change of scale property, obtain the Laplace transform of

(i) $\sinh 3t$ (ii) $\cos 5t$

Sol'n: (i) we have $L\{\sinh t\} = \frac{1}{P^2-1}$

$= f(P)$, say

$\therefore L\{\sinh 3t\} = \frac{1}{3} f(P/3)$ $\because L(F(t)) = f(P)$

$L\{F(at)\} = \frac{1}{a} f\left(\frac{P}{a}\right)$

$$= \frac{1}{3} \frac{1}{(P/3)^2 - 1}$$

$$= \frac{3}{P^2 - 9}$$

→ Given $L\{F(t)\} = \frac{P^2 - P + 1}{(2P+1)^2 (P-1)}$; applying the change of scale property show that

$$L\{F(2t)\} = \frac{P^2 - 2P + 4}{4(P+1)^2 (P-2)}$$

→ Find $L\{G(t)\}$, where $G(t) = \begin{cases} e^{t-a}, & t > a \\ 0, & t < a \end{cases}$

Sol'n: From second shifting theorem we know that if $L\{F(t)\} = f(P)$

and $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$

then $L\{G(t)\} = e^{-aP} f(P)$.

Here let $F(t) = e^t$

$$\therefore L\{F(t)\} = L\{e^t\} = \int_0^\infty e^{-Pt} e^t \, dt$$

$$= \int_0^\infty e^{-(P-1)t} \, dt$$

$$= \left[ \frac{e^{-(P-1)t}}{-(P-1)} \right]_0^\infty$$

$$= \frac{1}{P-1}, \quad P > 1$$

$$= f(P), \text{ say}$$

and $G(t) = \begin{cases} F(t-a) = e^{t-a}, & t > a \\ 0, & t < a \end{cases}$

$$\therefore L\{G(t)\} = e^{-aP} f(P)$$

$$= \frac{e^{-aP}}{P-1}, \quad P > 1$$

(OR)

we have $L\{G(t)\} = \int_0^\infty e^{-Pt} G(t) \, dt$

$$= \int_0^a e^{-Pt} G(t) \, dt + \int_a^\infty e^{-Pt} G(t) \, dt$$

$$= \int_0^a e^{-Pt} (0) \, dt + \int_a^\infty e^{-Pt} e^{t-a} \, dt$$

$$= e^{-a} \int_a^\infty e^{-(P-1)t} \, dt$$

$$= e^{-a} \left[ \frac{e^{-(P-1)t}}{-(P-1)} \right]_a^\infty$$

$$= e^{-a} \left[ 0 - \frac{e^{-(P-1)a}}{-(P-1)} \right]$$

$$= \frac{e^{-aP}}{P-1}$$

→ Find $L\{F(t)\}$, where

$$F(t) = \begin{cases} \cos\left(t - \frac{2}{3}\pi\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$$

→ Find $L\{G(t)\}$, where

$$F(t) = \begin{cases} \sin\left(t - \frac{\pi}{3}\right), & t > \frac{\pi}{3} \\ 0, & t < \frac{\pi}{3} \end{cases}$$

## *Laplace Transform of Derivatives:

**Theorem:** Let $F(t)$ be continuous for all $t \geq 0$ and be of exponential order 'a' as $t \to \infty$ and if $F'(t)$ is of class A, then Laplace transform of the derivative $F'(t)$ exists when $p > a$, and $L\{F'(t)\} = pL\{F(t)\} - F(0)$.

**Proof:**

**Case(i)** $F'(t)$ is continuous for all $t \geq 0$, then

$$L\{F'(t)\} = \int_0^\infty e^{-pt} F'(t)\, dt \quad \text{---} (1)$$

$$= \left[e^{-pt} F(t)\right]_0^\infty + \int_0^\infty p e^{-pt} F(t) dt$$

(Integrating by parts)

$$= \lim_{t\to\infty} e^{-pt} F(t) - F(0) + p\int_0^\infty e^{-pt} F(t) dt$$

$$L\{F'(t)\} = \lim_{t\to\infty} e^{-pt} F(t) - F(0) + pL\{F(t)\} \quad \text{---} (2).$$

Since $F(t)$ is continuous for all $t \geq 0$ and is of exponential order 'a' as $t \to \infty$.

$|F(t)| \leq Me^{at}$ $\forall$ $t \geq 0$ and for some constants $a$ and $M$,

we have $|e^{-pt} F(t)| = e^{-pt} |F(t)|$

$$\leq e^{-pt} Me^{at}$$

$$= Me^{-(p-a)t} \to 0$$

as $t \to \infty$ if $p > a$

$\therefore \lim_{t\to\infty} e^{-pt} F(t) = 0$ for $p > a$

$\therefore$ from (2) we conclude that $L\{F'(t)\}$ exists and $L\{F'(t)\} = pL\{F(t)\} - F(0)$.

**Case-2:** $F'(t)$ is merely piecewise continuous, the integral (1) may be broken as the sum of integrals in different ranges from 0 to $\infty$ such that $F'(t)$ is continuous in each of such parts.

Then proceeding as in case(i), we get

$$\boxed{L\{F'(t)\} = pL\{F(t)\} - F(0)}$$

$$L\{F'(t)\} = \int_0^\infty e^{-pt} F'(t)$$

$$= \int_0^{t_0} e^{-pt} F'(t) + \int_{t_0}^{t_1} e^{-pt} F'(t) dt + \cdots$$

$$+ \int_{t_0}^\infty e^{-pt} F'(t)\, dt,$$ [In particular $LF'(t) = \int_0^{t_0} + \int_{t_0}^\infty$]

$$= \left[e^{-pt} F(t)\right]_0^{t_0} + p\int_0^{t_0} e^{-pt} F(t) dt + \cdots$$

$$+ \left[e^{-pt} F(t)\right]_{t_0}^\infty + p\int_{t_0}^\infty e^{-pt} F(t) dt$$

$$= e^{-pt_0} \cancel{F(t_0)} - F(0) + \lim_{t\to\infty} e^{-pt} F(t) - \cancel{e^{-pt_0} F(t_0)}$$

$$+ p\int_0^{t_0} e^{-pt} F(t) dt + p\int_{t_0}^\infty e^{-pt} F(t) dt$$

$$= -F(0) + p\int_0^\infty e^{-p(t)} F(t)\, dt$$

$$= pL\{F(t)\} - F(0)$$

**Note 1:** If $F(t)$ fails to be continuous at $t=0$ but $\lim_{t\to 0+} F(t) = F(0+0)$ exists
$= F(0+)$

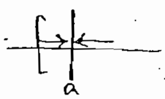
[i.e, $F(0+0)$ is not equal to $F(0)$, which may or may not exist]

then $L\{F'(t)\} = pL\{F(t)\} - F(0+0)$

**Note 2:** If $F(t)$ fails to be continuous at $t=a$, then

$$L\{F'(t)\} = pL\{F(t)\} - F(0) - e^{-ap}[F(a+0) - F(a-0)]$$

where $F(a+0)$ and $F(a-0)$ are the limits of $F$ at $t=a$, as $t$ approaches 'a' from the right and from the left respectively.



The quantity $F(a+0) - F(a-0)$ is called the jump discontinuity at $t=a$.

**Proof:** $L\{F'(t)\} = \int_0^\infty e^{-pt} F'(t)\,dt$

$$= \int_0^a e^{-pt}F'(t)\,dt + \int_a^\infty e^{-pt}F'(t)\,dt$$

$$= \left[e^{-pt}F(t)\right]_0^a + p\int_0^a e^{-pt}F(t)\,dt + \left[e^{-pt}F(t)\right]_a^\infty + p\int_a^\infty e^{-pt}F(t)\,dt$$

$$= e^{-pa}F(a-0) - F(0) + p\int_0^a e^{-pt}F(t)\,dt + \lim_{t\to\infty} e^{-pt}F(t) - e^{-pa}F(a+0) + p\int_a^\infty e^{-pt}F(t)\,dt$$

$\left(\because \lim_{t\to\infty} e^{-pt}F(t) = 0 \text{ by case (1)}\right)$

$$= e^{-ap}F(a-0) - e^{-pa}F(a+0) + 0 - F(0) + p\int_0^\infty e^{-pt}F(t)\,dt$$

$$= e^{-ap}[F(a-0) - F(a+0)] - F(0) + pL\{F(t)\}$$

$$= pL\{F(t)\} - F(0) + e^{-ap}[F(a-0) - F(a+0)]$$

**Note (3):** For more than one discontinuity of the function $F(t)$, appropriate modification can be made.

## * Laplace Transform of the $n^{th}$ order derivative of $F(t)$:

Let $F(t)$ and its derivatives $F'(t), F''(t), \ldots, F^{n-1}(t)$ be continuous functions for all $t \geq 0$ and be of exponential orders as $t \to \infty$ and if $F^n(t)$ is of class A, then laplace transform of $F^n(t)$ exists when $p > a$, and is given by

$$L\{F^n(t)\} = p^n L\{F(t)\} - p^{n-1}F(0) - p^{n-2}F'(0) - \cdots - F^{n-1}(0)$$

**Proof:** From the above theorem, we have

$$L\{F'(t)\} = pL\{F(t)\} - F(0) \quad \text{---(1)}$$

Applying the result (1) to the $2^{nd}$ order derivative $F''(t)$.

$$L\{F''(t)\} = pL\{F'(t)\} - F'(0)$$
$$= p[pL\{F(t)\} - F(0)] - F'(0) \quad (\because \text{by} ①)$$
$$= p^2L\{F(t)\} - pF(0) - F'(0) \quad \text{—— ②}$$

Again applying (1) to the 3rd order derivative $F'''(t)$, we have

$$L\{F'''(t)\} = pL\{F''(t)\} - F''(0)$$
$$= p[p^2L\{F(t)\} - pF(0) - F'(0)] - F''(0) \quad (\because \text{by} ②)$$
$$= p^3L\{F(t)\} - p^2F(0) - pF'(0) - F''(0)$$

proceeding similarly, we get

$$L\{F^{n}(t)\} = p^nL\{F(t)\} - p^{n-1}F(0) - p^{n-2}F'(0) - \cdots - F^{n-1}(0).$$

## * Initial Value Theorem:

Let $F(t)$ be continuous for all $t \geq 0$ and be of exponential order as $t \to \infty$ and if $F'(t)$ is of class A, then $\lim_{t\to 0} F(t) = \lim_{p\to\infty} pL\{F(t)\}$.

**Proof:** we know that

$$L\{F'(t)\} = pL\{F(t)\} - F(0)$$

$$\Rightarrow \int_0^\infty e^{-pt} F'(t)\,dt = pL\{F(t)\} - F(0) \quad \text{—— (1)}$$

Taking limit as $p \to \infty$ in (1)

$$\lim_{p\to\infty} \int_0^\infty e^{-pt} F'(t)\,dt = \lim_{p\to\infty} [pL\{F(t)\} - F(0)] \quad \text{—— ②}$$

[Since $F'(t)$ is of class A, i.e, $F'(t)$ is sectionally continuous and of exponential order.

$$\therefore \lim_{p\to\infty} \int_0^\infty e^{-pt} F'(t)\,dt = 0$$

$\therefore$ from ②

$$0 = \lim_{p\to\infty} pL\{F(t)\} - F(0)$$

$$\Rightarrow \lim_{p\to\infty} pL\{F(t)\} = F(0)$$

$$\Rightarrow \lim_{p\to\infty} pL\{F(t)\} = \lim_{t\to 0} F(t)$$

(or) $\lim_{t\to 0} F(t) = \lim_{p\to\infty} pL\{F(t)\}$

## → Final Value Theorem:

Let $F(t)$ be continuous for all $t \geq 0$ and be of exponential order as $t \to \infty$ and if $F'(t)$ is of class A, then $\lim_{t\to\infty} F(t) = \lim_{p\to 0} pL\{F(t)\}$

**Proof:** we know that

$$L\{F'(t)\} = pL\{F(t)\} - F(0)$$

$$\Rightarrow \int_0^\infty e^{-pt} F'(t)\,dt = pL\{F(t)\} - F(0) \quad \text{—— ①}$$

Taking limit as $p \to 0$ in (1), we get,

$$\lim_{p\to 0} \int_0^\infty e^{-pt} F'(t)\,dt = \lim_{p\to 0} [pL\{F(t)\} - F(0)]$$

$$\Rightarrow \int_0^{\infty} F'(t)dt = \lim_{P\to 0} PL\{F(t)\} - F(0)$$

$$\Rightarrow [F(t)]_0^{\infty} = \lim_{P\to 0} PL\{F(t)\} - F(0)$$

$$\Rightarrow \lim_{t\to\infty} F(t) - F(0) = \lim_{P\to 0} PL\{F(t)\} - F(0)$$

$$\Rightarrow \lim_{t\to\infty} F(t) = \lim_{P\to 0} PL\{F(t)\}$$

## * → Laplace – Transform of Integrals:

**Theorem:** If $F(t)$ is piecewise continuous and satisfies $|F(t)| \le Me^{at}$ for all $t \ge 0$ for some constants 'a' and M, then

$$L\left\{\int_0^t F(x)\,dx\right\} = \frac{1}{P} L\{F(t)\} = \frac{1}{P} f(P).$$

$(P>0,\ P>a)$

**Proof:** Let $F(t)$ be piecewise continuous such that

$|F(t)| \le Me^{at}$ for some constants —(1) a & M.

If (1) holds for some negative values of 'a' then it also holds for positive value of 'a'

∴ Suppose that a is +ve.

Let $G(t) = \int_0^t F(x)\,dx$

$G(t)$ is continuous and

$$|G(t)| = \left|\int_0^t F(x)\,dx\right|$$

$$\le \int_0^t |F(x)|\,dx$$

$$\le \int_0^t Me^{ax}\,dx \quad (\because \text{by } (1))$$

$$= \frac{M(e^{at}-1)}{a}, \ a>0$$

$$\therefore |G(t)| \le \frac{M}{a}(e^{at}-1), \ a>0.$$

Also $G'(t) = F(t)$, except for points at which $F(t)$ is discontinuous.

$G'(t) = \frac{d}{dt}\int_0^t F(x)dx$ by Leibnitz rule

$= \int_0^t \frac{d}{dt}F(x)dx + F(x)\Big|_{x=t}\frac{dt}{dt} - F(x)\Big|_{x=0}\frac{d0}{dt}$

$= 0 + F(t) - 0 = F(t)$

∴ $G'(t)$ is piecewise continuous on each finite interval and is of exponential order.

∴ By existence theorem, laplace transform of $G'(t)$ exists.

$$\therefore L(G'(t)) = PL\{G(t)\} - G(0)$$

$$= PL\{G(t)\} \quad \left(\because G(0) = \int_0^0 F(x)dx \Rightarrow G(0) = 0\right)$$

$$\Rightarrow L\{G(t)\} = \frac{1}{P} L(G'(t))$$

$$\Rightarrow L\left\{\int_0^t F(x)dx\right\} = \frac{1}{P} L\{F(t)\}$$

## * Multiplication by powers of t:

### Multiplication by t

**Theorem:** If $F(t)$ is a function of class A and if $L\{F(t)\}=f(P)$ then

$$L\{tF(t)\}=-f'(P).$$

**Proof:** We have

$$f(P)=L\{F(t)\}=\int_0^\infty e^{-Pt}F(t)\,dt$$

$$\therefore f'(P)=\frac{d}{dP}\int_0^\infty e^{-Pt}F(t)\,dt$$

$$=\int_0^\infty \frac{\partial}{\partial P}\{e^{-Pt}F(t)\,dt\}+0-0 \quad \text{(By Leibnitz's rule)}$$

$$=-\int_0^\infty te^{-Pt}F(t)\,dt$$

$$=-\int_0^\infty e^{-Pt}\{tF(t)\}\,dt$$

$$=-L\{tF(t)\}$$

$$\therefore L\{tF(t)\}=-f'(P)$$

(or)

$$L\{tF(t)\}=(-1)^1\frac{d}{dP}f(P)$$

Differentiation of a function under an integral sign.

$$\frac{d}{dx}\int_A^B F(x,t)\,dt=\int_A^B \frac{\partial}{\partial x}\{F(x,t)\}\,dt+F(x,t)\Big|_{x=B}\frac{dB}{dx}-F(x,t)\Big|_{x=A}\frac{dA}{dx}$$

where A & B are functions of x or constants.

### Multiplication by $t^n$:

If $F(t)$ is a function of class A and if $L\{F(t)\}=f(P)$ then

$$L\{t^nF(t)\}=(-1)^n\frac{d^n}{dP^n}f(P),$$

where $n=1,2,3\ldots$

### Division by t:

**Theorem:** If $L\{F(t)\}=f(P)$, then

$$L\left\{\frac{1}{t}F(t)\right\}=\int_P^\infty f(x)\,dx$$

provided $\lim_{t\to 0}\left\{\frac{1}{t}F(t)\right\}$ exists.

**Proof:** Let $G(t)=\frac{1}{t}F(t)$

$$\Rightarrow F(t)=t\,G(t)$$

Apply L.T. on bothsides

$$L\{F(t)\}=L\{tG(t)\}$$

$$\Rightarrow f(P)=-\frac{d}{dP}L\{G(t)\} \qquad \boxed{\because L\{tF(t)\}=-f'(P)}$$

Integrating bothsides w.r.t P from P to $\infty$, we get

$$\int_P^\infty f(P)\,dP=-\Big[L\{G(t)\}\Big]_P^\infty=-\left[\int_0^\infty e^{-Pt}G(t)\,dt\right]_{P=P}^{\infty}$$

$$\Rightarrow \int_P^\infty f(P)\,dP=-\lim_{P\to\infty}\left[\int_0^\infty e^{-Pt}G(t)\,dt\right]+\lim_{P\to P}\left[\int_0^\infty e^{-Pt}G(t)\,dt\right]$$

$$=0+\int_0^\infty e^{-Pt}G(t)\,dt \qquad \left[\because \lim_{P\to\infty}L\{G(t)\}=\lim_{P\to\infty}\int_0^\infty e^{-Pt}G(t)\,dt=0\right]$$

$$=L\{G(t)\}$$

$\Rightarrow L(G(t)) = \int_P^\infty f(P)dp = \int_P^\infty f(x)dx$ ($\because$ By property of definite integral $\int_a^b f(x)dx = \int_a^b f(t)dt$)

i.e,

$$\boxed{L\left\{\frac{1}{t}F(t)\right\} = \int_P^\infty f(x)dx.}$$

Problems:

$\rightarrow$ Using the Laplace transform of Derivative of $F(t)$ i.e,

$$L(F^n(t)) = P^n L\{F(t)\} - P^{n-1}F(0) - P^{n-2}F'(0) - \cdots - P^{n-1}(0).$$

Show that (1) $L\{t\} = \frac{1}{P^2}$

(2) $L\{e^{at}\} = \frac{1}{P-a}$ (3) $L\{-a\sin at\} = \frac{-a^2}{P^2+a^2}$

Sol'n: (1) we have

$$L\{F'(t)\} = PL\{F(t)\} - F(0) \quad \text{--- (1)}$$

Here let $F(t) = t$ then $F'(t) = 1$ and $F(0) = 0$

$\therefore$ from (1)

$$L\{1\} = PL\{t\} - 0$$

$$L\{t\} = \frac{1}{P}L\{1\}$$

$$= \frac{1}{P}\cdot\frac{1}{P} \quad (\because L\{1\} = \frac{1}{P})$$

$$= \frac{1}{P^2}, \; P>0$$

(2) Let $F(t) = e^{at}$

then $F'(t) = ae^{at}$ and $F(0) = 1$

$\therefore$ from (1)

$$L\{ae^{at}\} = PL\{e^{at}\} - 1$$

$$\Rightarrow aL\{e^{at}\} = PL\{e^{at}\} - 1$$

$$\Rightarrow (P-a)L\{e^{at}\} = 1$$

$$\Rightarrow L\{e^{at}\} = \frac{1}{P-a}$$

(3) Let $F(t) = -a\sin at$

$F'(t) = -a^2\cos at$ and $F''(t) = a^3\sin at$

Also $F(0) = 0$ and $F'(0) = -a^2$

we know that

$$L\{F''(t)\} = P^2L\{F(t)\} - PF(0) - F'(0)$$

$$\Rightarrow L\{a^3\sin at\} = P^2L\{-a\sin at\} - P(0) - (-a^2)$$

$$\Rightarrow a^2L\{a\sin at\} = P^2L\{-a\sin at\} + a^2$$

$$\Rightarrow P^2L\{-a\sin at\} - a^2L\{a\sin at\} = -a^2$$

$$\Rightarrow P^2L\{-a\sin at\} + a^2L\{-a\sin at\} = -a^2$$

$$\Rightarrow (P^2+a^2)L\{-a\sin at\} = -a^2$$

$$\Rightarrow L\{-a\sin at\} = \frac{-a^2}{P^2+a^2} \quad \text{--- (2)}$$

Note: from equation (2)

$$\boxed{L\{\sin at\} = \frac{a}{P^2+a^2}}$$

$\rightarrow$ Find $L\{t\cos at\}$. It is in the form of $L\{t\,F(t)\}$.

Sol'n: Since $L\{\cos at\} = \frac{P}{P^2+a^2} = (\text{say } f(P))$, $P>0$

$\therefore L\{t\cos at\} = -\frac{d}{dp} L\{\cos at\}$

$= -\frac{d}{dp}\left(\frac{p}{p^2+a^2}\right)$ [$\because$ if $L\{F(t)\}=f(p)$ then $L\{tF(t)\} = -\frac{d}{dp}f(p)$]

$= -\frac{p^2+a^2-p(2p)}{(p^2+a^2)^2}$

$= -\frac{a^2-p^2}{(p^2+a^2)^2} = \frac{p^2-a^2}{(p^2+a^2)^2}$

→ Find (i) $L\{t^2\sin at\}$ (ii) $L\{t^2e^{2t}\}$
(iii) $L\{t^3\cos t\}$ (iv) $L\{t(3\sin 2t - 2\cos 2t)\}$
(v) $L\{\sin at - at\cos at\}$

→ show that

$L\{t^n e^{at}\} = \frac{n!}{(p-a)^{n+1}}, \quad p>a$

Sol'n: Since $L\{e^{at}\} = \frac{1}{p-a}$ (say $f(p)$), $\therefore p>a$

$\therefore L\{t^n e^{at}\} = (-1)^n \frac{d^n}{dp^n} f(p)$

$= (-1)^n \frac{d^n}{dp^n}\left(\frac{1}{p-a}\right)$

$= (-1)^n \frac{(-1)^n n!}{(p-a)^{n+1}}$

$= \frac{n!}{(p-a)^{n+1}}, \quad p>a$

→ show that $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{p}}\, e^{-1/4p}$

Sol'n: Let $F(t) = \sin\sqrt{t}$

then $F'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$ and $F(0)=0$

$\therefore$ From $L\{F'(t)\} = pL\{F(t)\} - F(0)$

$\Rightarrow L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = pL\{\sin\sqrt{t}\} - 0$

$\Rightarrow L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = 2p\, L\{\sin\sqrt{t}\}$

$= 2p\,\frac{\sqrt{\pi}}{2p^{3/2}}\, e^{-1/4p}$ [w.k.t $L(\sin\sqrt{t}) = \frac{\sqrt{\pi}}{2p^{3/2}} e^{-1/4p}$]

$= \frac{\sqrt{\pi}}{\sqrt{p}}\, e^{-1/4p}$

$= \sqrt{\frac{\pi}{p}}\, e^{-1/4p}$

→ Prove that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{p}\right)$ and hence find $L\left\{\frac{\sin at}{t}\right\}$. Does the Laplace transform of $\frac{\cos at}{t}$ exist?

Sol'n: Let $F(t) = \sin t$ [Given problem is in the form of $L\left\{\frac{F(t)}{t}\right\}$ check $\lim_{t\to 0}\frac{F(t)}{t}$ exist or not.]

Now $\lim_{t\to 0}\frac{F(t)}{t} = \lim_{t\to 0}\frac{\sin t}{t}$

$= 1$

Since $L\{\sin t\} = \frac{1}{p^2+1} = f(p)$, say

$\therefore$ from $L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(x)\,dx$, we have

$L\left\{\frac{\sin t}{t}\right\} = \int_p^\infty \frac{1}{x^2+1}\,dx$ [$\because f(p) = \frac{1}{p^2+1} \Rightarrow f(x) = \frac{1}{x^2+1}$]

$= \left[\tan^{-1}x\right]_p^\infty$ [$\because$ If $x\in R$, $x\neq 0$ then $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2} \Rightarrow \frac{\pi}{2} - \tan^{-1}x = \cot^{-1}x$]

$= \left[\frac{\pi}{2} - \tan^{-1}p\right]$

$= \cot^{-1}p$

$= \tan^{-1}\frac{1}{p}$ $\left[\because \text{Let } \cot^{-1}p = x \Rightarrow p = \cot x \Rightarrow \tan x = 1/p \Rightarrow x = \tan^{-1}(1/p)\right]$

$= f_1(p)$ say

Now $L\left\{\frac{\sin at}{t}\right\} = aL\left\{\frac{\sin(at)}{at}\right\}$

$= a\cdot\frac{1}{a}\tan^{-1}\frac{1}{(p/a)}$

$= \tan^{-1}\frac{a}{p}$ $\left[\because L\{F(at)\} = \frac{1}{a}f_1(p/a)\right]$

Since $L\{\cos at\} = \frac{p}{p^2+a^2} = f(p)$, say

we have $L\left\{\frac{\cos at}{t}\right\} = \int_p^\infty \frac{x}{x^2+a^2}dx$

$= \left[\frac{1}{2}\log(x^2+a^2)\right]_p^\infty$

$= \frac{1}{2}\lim_{x\to\infty}\log(x^2+a^2) - \frac{1}{2}\log(p^2+a^2)$

which does not exist.

$\left(\because \lim_{x\to\infty}\log(x^2+a^2) \text{ is infinite}\right)$

Hence $L\left\{\frac{\cos at}{t}\right\}$ does not exist.

→ If $L\{F(t), t\to p\} = f(p)$, show that

$L\left\{\int_0^t \frac{F(u)}{u}du, t\to p\right\} = \frac{1}{p}\int_p^\infty f(y)dy$

Hence show that

$L\left\{\int_0^t \frac{\sin u}{u}du, t\to p\right\} = \frac{\cot^{-1}p}{p}$

Sol'n: From the laplace transform of integrals

we know that $L\left\{\int_0^t F(u)du\right\} = \frac{1}{p}f(p)$ —— ①

where $f(p) = L\{F(t)\}$.

Let $G(t) = \frac{F(t)}{t}$

$\therefore L\{G(t)\} = L\left\{\frac{F(t)}{t}\right\}$ $\left[\because \text{Division by } L\left(\frac{F(t)}{t}\right) = \int_p^\infty f(x)dx \text{ if } L(F(t)) = f(p)\right]$

$= \int_p^\infty f(y)dy$

$= g(p)$, say

$\therefore$ from ①

$L\left\{\int_0^t G(u)du\right\} = \frac{1}{p}g(p)$

$\Rightarrow L\left\{\int_0^t \frac{F(u)}{u}du\right\} = \frac{1}{p}\int_p^\infty f(y)dy$ —— ②

Deduction:

Let $F(t) = \sin t$

So that $f(p) = L\{\sin t\} = \frac{1}{p^2+1}$

$\therefore$ from ②

$L\left\{\int_0^t \frac{\sin u}{u}du\right\} = \frac{1}{p}\int_p^\infty \frac{1}{y^2+1}dy$

$= \frac{1}{p}\left[\tan^{-1}y\right]_p^\infty$

$= \frac{1}{p}\left[\frac{\pi}{2} - \tan^{-1}p\right]$

$= \frac{1}{p}\cot^{-1}p$

→ Prove that $\int_0^\infty \frac{F(t)}{t}dt = \int_0^\infty f(x)dx$,

Provided that the integral converges.

Sol'n: we have $L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(x)dx$

(Division by $t$)

$$\Rightarrow \int_0^\infty e^{-pt}\frac{F(t)}{t}\,dt = \int_p^0 f(x)dx + \int_0^\infty f(x)dx$$

$$= -\int_0^p f(x)dx + \int_0^\infty f(x)dx$$

i.e. $\lim_{p\to 0^+}\int e^{-pt}\frac{F(t)}{t}dt$

$= \lim_{p\to 0^+}\left[-\int_0^p f(x)dx + \int_0^\infty f(x)dx\right]$

i.e. $-\int_0^0 f(x)dx + \int_0^\infty f(x)dx$

Taking limit as $p\to 0^+$, (assuming the integral converges)

we have

$$\int_0^\infty \frac{F(t)}{t}dt = \int_0^\infty f(x)dx$$

Show that $\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}$

Sol$^n$: Let $F(t) = \sin t$

$$f(p) = L\{F(t)\} = L\{\sin t\} = \frac{1}{p^2+1}$$

$$\therefore L\left\{\frac{\sin t}{t}\right\} = \int_0^\infty e^{-pt}\frac{\sin t}{t}dt$$

$$\left[\because L\left\{\frac{\sin t}{t}\right\} = \int_p^\infty f(x)dx\right]$$

$$= \int_p^\infty f(x)dx$$

$$= \int_p^\infty \frac{1}{1+x^2}dx$$

$$= \left[\tan^{-1}(x)\right]_p^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}p$$

Taking limit as $p\to 0$,

we have $\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}$

## Evaluation of Integrals:

If $L\{F(t)\} = f(p)$

i.e, $\int_0^\infty e^{-pt}F(t)dt = f(p)$

taking limit as $p\to 0$, we have

$$\int_0^\infty F(t)dt = f(0)$$

assuming that the integral is convergent.

Evaluate $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}dt$

Sol$^n$: Let $F(t) = e^{-at} - e^{-bt}$

$$\therefore f(p) = L\{F(t)\} = L\{e^{-at}\} - L\{e^{-bt}\}$$

$$= \frac{1}{p+a} - \frac{1}{p+b}$$

$$\therefore L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(x)dx$$

$$\Rightarrow \int_0^\infty e^{-pt}\left(\frac{e^{-at}-e^{-bt}}{t}\right)dt = \int_p^\infty\left(\frac{1}{x+a} - \frac{1}{x+b}\right)dx$$

$$= \left[\log(x+a) - \log(x+b)\right]_p^\infty$$

$$= \log\left(\frac{1+\frac{a}{x}}{1+\frac{b}{x}}\right)_p^\infty$$

$$= 0 - \left[\log(p+a) - \log(p+b)\right]$$

$$= \log\left(\frac{p+b}{p+a}\right)$$

Taking limit as $p\to 0$, we have

$$\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}dt = \log\frac{b}{a}.$$

$\rightarrow$ show that $\int_0^\infty \frac{e^{-3t}-e^{-6t}}{t}\,dt = \log 2$

$\rightarrow$ Evaluate $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt$

$\rightarrow$ Show that $\int_0^\infty te^{-3t}\sin t\,dt = \frac{3}{50}$

Sol'n: Given $\int_0^\infty te^{-3t}\sin t\,dt =$

$$= \int_0^\infty e^{-3t}(t\sin t)\,dt.$$

which is in the form of

$\int_0^\infty e^{-pt}(t\sin t)\,dt = L(t\sin t)$ where $p=3$

Since $L\{\underset{F(t)}{\sin t}\} = \frac{1}{p^2+1} = f(p)$ say

and $L\{t\sin t\} = (-1)\frac{d}{dp}\frac{1}{p^2+1}$

$(\because L\{t\,F(t)\} = (-1)f'(p))$

(or) $\int_0^\infty e^{-pt}\,t\sin t\,dt = (-1)\frac{-2p}{(p^2+1)^2}$

putting $p=3$, we have

$$\int_0^\infty e^{-3t}\,t\sin t\,dt = \frac{3}{50}$$

$$\Rightarrow \int_0^\infty te^{-3t}\sin t\,dt = \frac{3}{50}$$

$\rightarrow$ show that $\int_0^\infty te^{-2t}\cos t\,dt = \frac{3}{25}$

Sol'n: we have $L\{t\cos t\} = -\frac{d}{dp}L\{\cos t\}$

$$\Rightarrow \int_0^\infty e^{-pt}\,t\cos t\,dt = \frac{-d}{dp}\left(\frac{p}{p^2+1}\right)$$

$$= -\frac{[p^2+1-p(2p)]}{(p^2+1)^2}$$

$$= \frac{p^2-1}{(p^2+1)^2}$$

Putting $p=2$, we get

$$\int_0^\infty te^{-2t}\cos t\,dt = \frac{3}{25}$$

$\rightarrow$ Prove that $\int_0^\infty t^3e^{-t}\sin t\,dt = 0$.

## *Periodic Functions:

Let $F$ be a periodic function with period $T>0$, that is $F(u+T) = F(u)$, $F(u+2T) = F(u)$ etc. then

$$L\{F(t)\} = \frac{\int_0^T e^{-pt}F(t)\,dt}{1-e^{-pT}}$$

Proof: we have $L\{F(t)\} = \int_0^\infty e^{-pt}F(t)\,dt$

$$= \int_0^T e^{-pt}F(t)\,dt + \int_T^{2T} e^{-pt}F(t)\,dt + \int_{2T}^{3T} e^{-pt}F(t)\,dt + \cdots \quad (1)$$

Putting $t=u+T$ in 2nd integral; $t=u+2T$ in 3rd integral & so on.

$\Rightarrow dt = du$ ; $\Rightarrow dt = du$

Limits: If $t=T \Rightarrow u=0$, $t=2T \Rightarrow u=T$ ; if $t=2T \Rightarrow u=0$, if $t=3T \Rightarrow u=T$

$\therefore$ from (1)

$$L\{F(t)\} = \int_0^T e^{-pt}F(t)\,dt + \int_0^T e^{-p(u+T)}F(u+T)\,du + \int_0^T e^{-p(u+2T)}F(u+2T)\,du + \cdots$$

$$\boxed{\begin{array}{l}\text{In the 1st integral}\\ \int_0^T e^{-pt}F(t)dt\\ =\int_0^T e^{-pu}F(u)du\end{array}}$$

$$= \int_0^T e^{-pu}F(u)\,du + e^{-pT}\int_0^T e^{-pu}F(u)\,du + e^{-2pT}\int_0^T e^{-pu}F(u)\,du + \cdots$$

$$= \underbrace{[1+e^{-pT}+e^{-2pT}+\cdots]}_{[(1-x)^{-1}]}\int_0^T e^{-pu}F(u)\,du$$

* A real function $f: A \to R$ is said to be a periodic function if $\exists$ a +ve real no. P such that $f(x+P) = f(x), \forall x \in A$. The least +ve real number P such that $f(x+P) = \cancel{f(x), \forall x \in A}$ is called period of f.

Ex: $f(x) = \sin x$ is a periodic function with period $2\pi$.

→ for $\cos x$ is $2\pi$, → for $\tan x$ is $\pi$.

## * Some Special Functions:

### → The Sine and Cosine Integrals:

The sine and cosine integrals, denoted by $S_i(t)$ and $C_i(t)$ respectively are defined by

$$S_i(t) = \int_0^t \frac{\sin u}{u} du$$

and $$C_i(t) = \int_t^\infty \frac{\cos u}{u} du$$

### → The Error Function:

The Error function, denoted by $erf(t)$, is defined by

$$erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du$$

### → The Gamma Function:

If $n > 0$, the Gamma function is defined by

$$\Gamma(n) = \int_0^\infty u^{n-1} e^{-u} du$$

### → The unit step Function (also called Heaviside's Unit Function):

The unit step function, denoted by $H(t-a)$ is defined by.

$$H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a. \end{cases}$$

### → The Exponential Integral:

The exponential integral is defined by $$E_i(t) = \int_t^\infty \frac{e^{-u}}{u} du.$$

### → * The Bessel Function:

Bessel Function of order n is defined by

$$J_n(t) = \frac{t^n}{2^n \Gamma(n+1)} \left[ 1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2 \cdot 4 (2n+2)(2n+4)} \right]$$

$$= \sum_{r=0}^\infty \frac{(-1)^r}{r! \, \Gamma(n+r+1)} \left(\frac{t}{2}\right)^{n+2r}$$

$$\left\{ J_n(t) = \frac{(-1)^0}{\Gamma n+1} \left(\frac{t}{2}\right)^n + \frac{(-1)^1}{1! \, \Gamma n+2} \left(\frac{t}{2}\right)^{n+2} \right.$$

$$+ \frac{(-1)^2}{2! \, \Gamma n+3} \left(\frac{t}{2}\right)^{n+4} + \cdots$$

$$= \frac{1}{\Gamma n+1} \cdot \frac{t^n}{2^n} + \frac{1}{(n+1)\Gamma n+1} \frac{t^2}{2^2 \cdot 2^n} +$$

$$\frac{1}{2(n+2)(n+1)\Gamma n+1} \frac{t^n}{2^n 2^4} + \cdots$$

$$= \frac{t^n}{2^n \Gamma(n+1)} \left[ 1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2 \cdot 2^3 (2n+2)(2n+4)} - \cdots \right] \bigg\}$$

### →* Laguerre Polynomial

Laguerre polynomial is defined by

$$L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} (e^{-t} t^n), \; n = 0, 1, 2 \cdots$$

→ show that

(i) $L\{\sinh at \cos at\} = \dfrac{a(p^2-2a^2)}{p^4+4a^4}$

and (ii) $L\{\sinh at \sin at\} = \dfrac{2a^2 p}{p^4+4a^4}$

Sol'n: Since $L\{\sinh at\} = \dfrac{a}{p^2-a^2} = f(p)$, say

$\therefore L\{e^{iat}\sinh at\} = f(p-ia)$ (by first shifting property)

$$= \frac{a}{(p-ia)^2-a^2}$$

$$= \frac{a}{(p^2-2a^2)-2iap}$$

$(p-ia)^2-a^2 = p^2-2iap-a^2-a^2 = p^2-2a^2-2iap$

$$= \frac{a\{(p^2-2a^2)+2iap\}}{(p^2-2a^2)^2+4a^2p^2}$$

($\because$ by dividing and multiplying by $(p^2-2a^2)+2iap$)

$$= \frac{a(p^2-2a^2)+2ia^2p}{p^4-\cancel{4a^2p^2}+4a^4+\cancel{4a^2p^2}}$$

$$= \frac{a(p^2-2a^2)+2ia^2p}{p^4+4a^4}$$

$\Rightarrow L\{\sinh at(\cos at + i\sin at)\}$

$$= \frac{a(p^2-2a^2)}{p^4+4a^4} + i\frac{2a^2p}{p^4+4a^4}$$

$\Rightarrow L\{\sinh at \cos at\} + iL\{\sinh at \sin at\}$

$$= \frac{a(p^2-2a^2)}{p^4+2a^4} + i\frac{2a^2p}{p^4+4a^4}$$

Comparing real and imaginary terms on bothsides, we get

$$L\{\sinh at \cos at\} = \frac{a(p^2-2a^2)}{p^4+4a^4}$$

and $$L\{\sinh at \sin at\} = \frac{2a^2p}{p^4+4a^4}$$

→ Prove that $L\{J_0(t)\} = \dfrac{1}{\sqrt{1+p^2}}$

and hence deduce that (i) $L\{J_0(at)\} = \dfrac{1}{\sqrt{p^2+a^2}}$

(ii) $L\{t J_0(at)\} = \dfrac{p}{(p^2+a^2)^{3/2}}$

(iii) $L\{e^{-at} J_0(at)\} = \dfrac{1}{p^2+2ap+2a^2}$

(iv) $\displaystyle\int_0^\infty J_0(t)\,dt = 1$

Sol'n: we have

$$J_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{t}{2}\right)^{n+2r}$$

if $n=0$

$$J_0(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+1)}\left(\frac{t}{2}\right)^{2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,r!}\left(\frac{t}{2}\right)^{2r} = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2}\left(\frac{t}{2}\right)^{2r}$$

$$= 1 - \frac{t^2}{2^2} + \frac{t^4}{(2!)^2 2^4} - \frac{t^6}{(3!)^2 2^6} + \cdots$$

$$= 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2\cdot 4^2} - \frac{t^6}{2^2\cdot 4^2\cdot 6^2} + \cdots$$

$$L\{J_0(t)\} = L\{1\} - \frac{1}{2^2}L\{t^2\} + \frac{1}{2^2 \cdot 4^2}L\{t^4\} - \frac{1}{2^2 \cdot 4^2 \cdot 6^2}L\{t^6\} + \cdots$$

$$= \frac{1}{p} - \frac{1}{2^2}\cdot\frac{2!}{p^3} + \frac{1}{2^2 \cdot 4^2}\cdot\frac{4!}{p^5} - \frac{1}{2^2 \cdot 4^2 \cdot 6^2}\frac{6!}{p^7} + \cdots$$

$$= \frac{1}{p}\left[1 - \frac{1}{2p^2} + \frac{1}{2^2 \cdot 4^2}\frac{4\times3\times2\times1}{p^4} - \frac{1}{2^2 \cdot 4^2 \cdot 6^2}\frac{6\times5\times4\times3\times2\times1}{p^6} + \cdots\right]$$

$$= \frac{1}{p}\left[1 - \frac{1}{2}\left(\frac{1}{p^2}\right) + \frac{1.3}{2.4}\left(\frac{1}{p^2}\right)^2 - \frac{1.3.5}{2.4.6}\left(\frac{1}{p^2}\right)^3 + \cdots\right]$$

$$= \frac{1}{p}\left[1 + \frac{1}{p^2}\right]^{-1/2}$$

$$= \frac{1}{p\left(1+\frac{1}{p^2}\right)^{1/2}} = \frac{1}{\frac{p\sqrt{1+p^2}}{p}}$$

$$= \frac{1}{\sqrt{1+p^2}}$$

(i) we know that by change of scale property if $L\{F(t)\} = f(p)$, then

$$L\{F(at)\} = \frac{1}{a}f\left(\frac{p}{a}\right)$$

$$\therefore J_0(at) = \frac{1}{a}\frac{1}{\sqrt{1+\left(\frac{p}{a}\right)^2}} = \frac{1}{\sqrt{p^2+a^2}}$$

(ii) $L\{t\,J_0(at)\} = \frac{-d}{dp}L\{J_0(at)\}$.

$$\left[\because L\{tF(t)\} = \frac{-d}{dp}L\{F(t)\}\right]$$

$$= \frac{-d}{dp}\left(\frac{1}{\sqrt{p^2+a^2}}\right) = \frac{p}{(p^2+a^2)^{3/2}}$$

(iii) By first shifting theorem

$L\{e^{at}F(t)\} = f(p-a)$ if $L\{F(t)\} = f(p)$

$\Rightarrow L\{e^{-at}F(t)\} = f(p+a)$

$$\therefore L\{e^{-at}J_0(at)\} = \frac{p+a}{\sqrt{(p+a)^2+a^2}}$$

$$= \frac{p+a}{\sqrt{p^2+2ap+2a^2}} \qquad \left(\because L\{J_0(at)\} = \frac{1}{\sqrt{p^2+a^2}}\right)$$

(iv) we have $L\{J_0(at)\} = \frac{1}{\sqrt{p^2+a^2}}$

$$\Rightarrow \int_0^\infty e^{-pt}J_0(at)\,dt = \frac{1}{\sqrt{p^2+a^2}}$$

$$\Rightarrow \int_0^\infty e^{-pt}J_0(t)\,dt = \frac{1}{\sqrt{p^2+1}} \quad (\text{Putting } a=1)$$

$\therefore$ putting $p=0$, we get

$$\int_0^\infty J_0(t)\,dt = 1$$

→ Prove that $L\{J_1(t)\} = 1 - \frac{p}{\sqrt{p^2+1}}$ and hence deduce that $L\{t\,J_1(t)\} = \frac{1}{(p^2+1)^{3/2}}$

Sol'n: we know that

Hint $\quad J_n(t) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{t}{2}\right)^{n+2r}$

→ Find the Laplace Transform of $Si(t)$

Sol'n: we know that

$$Si(t) = \int_0^t \frac{\sin u}{u}\,du$$

$$= \int_0^t \left(1 - \frac{u^3}{3!} + \frac{u^4}{5!} - \frac{u^6}{7!} + \cdots\right)du$$

$$= t - \frac{t^3}{3(3!)} + \frac{t^5}{5(5!)} - \frac{t^7}{7(7!)} + \cdots$$

$$\therefore L\{Si(t)\} = L(t) - \frac{1}{3(3!)}L\{t^3\} + \frac{1}{5(5!)}L\{t^5\}$$

$$- \frac{1}{7(7!)}L\{t^7\} + \cdots$$

$$= \frac{1!}{p^2} - \frac{1}{3(3!)}\cdot\frac{3!}{p^4} + \frac{1}{5(5!)}\frac{5!}{p^6}$$

$$- \frac{1}{7(7!)}\frac{7!}{p^8} + \cdots$$

$$= \frac{1}{p}\left[\frac{1}{p} - \frac{1}{3}\frac{1}{p^3} + \frac{1}{5}\frac{1}{p^5} - \frac{1}{7}\cdot\frac{1}{p^7} + \cdots\right]$$

$$= \frac{1}{p}\tan^{-1}\frac{1}{p}$$ by Gregrey's series

→ Find $L\{Ci(t)\}$

Sol'n: $L\{Ci(t)\} = L\left\{\int_t^\infty \frac{\cos u}{u}\,du\right\}$

Let $F(t) = \int_t^\infty \frac{\cos u}{u}\,du = -\int_\infty^t \frac{\cos u}{u}\,du$

$$\Rightarrow F'(t) = -\frac{d}{dt}\int_\infty^t \frac{\cos u}{u}\,du$$

By Leibnitz's

So that $F'(t) = \frac{-\cos t}{t}$

(or) $tF'(t) = -\cos t$

$\therefore L\{tF'(t)\} = L\{-\cos t\}$

(or) $\frac{-d}{dp}L\{F'(t)\} = \frac{-p}{p^2+1}$

(or) $\frac{d}{dp}\left[pf(p) - F(0)\right] = \frac{p}{p^2+1}$ where $f(p) = L\{F(t)\}$

(or) $\frac{d}{dp}\left[pf(p)\right] = \frac{p}{p^2+1}$, since $F(0)$ is Constant.

Integrating $pf(p) = \frac{1}{2}\log(p^2+1) + c$ (Constant) ——— (i)

But from the final value theorem

$$\lim_{p\to 0} pf(p) = \lim_{t\to\infty} F(t) = 0$$

$\therefore$ from (i) as $p \to 0$, we have

$0 = 0 + c$ or $c = 0$

$\therefore$ from (i), $pf(p) = \frac{1}{2}\log(p^2+1)$

(or) $f(p) = L\{F(t)\} = L\{Ci(t)\} = \frac{\log(p^2+1)}{2p}$

→ If $F(t) = t^2,\ 0 < t < 2$ and $F(t+2) = F(t)$, find $L\{F(t)\}$

Sol'n: Here $F(t)$ is a periodic function with period $T = 2$.

$\therefore$ from fundamental theorem (Periodic function)

we have

$$L\{F(t)\} = \frac{\int_0^T e^{-pt} F(t)\,dt}{1-e^{-pT}}$$

$$= \frac{\int_0^2 t^2 e^{-pt}\,dt}{1-e^{-2p}}$$

$$= \frac{\left(-\frac{t^2}{p} e^{-pt}\right)_0^2 + \frac{2}{p}\int_0^2 t e^{-pt}\,dt}{1-e^{-2p}}$$

$$= \frac{-\frac{4}{p}e^{-2p} + \frac{2}{p}\left\{\left(-\frac{t}{p}e^{-pt}\right)_0^2 + \frac{1}{p}\int_0^2 e^{-pt}dt\right\}}{1-e^{-2p}}$$

$$= \frac{-\frac{4}{p}e^{-2p} - \frac{4}{p^2}e^{-2p} - \frac{2}{p^2}\left(\frac{-e^{-pt}}{p}\right)_0^2}{1-e^{-2p}}$$

$$= \frac{-(4p^2+4p+2)e^{-2p}+2}{p^3(1-e^{-2p})}$$

→ Find the Laplace transform of Heaviside's unit function $H(t-a)$

Sol'n: we have $H(t-a) = \begin{cases} 0, & t<a \\ 1, & t>a \end{cases}$

$$\therefore L\{H(t-a)\} = \int_0^\infty e^{-pt} H(t-a)\,dt$$

$$= \int_a^\infty e^{-pt} \cdot 1\,dt$$

$$= \frac{e^{-ap}}{p}.$$

→ show that $L\{E_i(t)\} = \frac{\log(p+1)}{p}$

Sol'n: we have $E_i(t) = \int_t^\infty \frac{e^{-u}}{u}\,du$

$$\therefore L\{E_i(t)\} = L\left\{\int_t^\infty \frac{e^{-u}}{u}\,du\right\}$$

$$= L\left\{\int_1^\infty \frac{e^{-tv}}{v}\,dv\right\}dt,$$

Putting $u = tv$ so that $du = t\,dv$

$$= \int_0^\infty e^{-pt}\left\{\int_1^\infty \frac{e^{-tv}}{v}\,dv\right\}dt$$

(By definition of L.T)

$$= \int_1^\infty \frac{1}{v}\left\{\int_0^\infty e^{-pt} e^{-tv}\,dt\right\}dv,$$

Changing the order of integration.

$$= \int_1^\infty \frac{1}{v}\,\frac{1}{p+v}\,dv$$

$$= \int_1^\infty \frac{1}{p}\left(\frac{1}{v} - \frac{1}{p+v}\right)dv$$

$$= \frac{1}{p}\Big[\log v - \log(p+v)\Big]_1^\infty$$

$$= \frac{1}{p}\left[-\log\left(\frac{p}{v}+1\right)\right]_1^\infty$$

$$= \frac{1}{p}\log(p+1)$$

→ Find $L\{L_n(t)\}$

we have $L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(e^{-t}t^n)$

$$\therefore L\{L_n(t)\} = \int_0^\infty e^{-pt} \cdot \frac{e^t}{n!}\frac{d^n}{dt^n}(e^{-t}t^n)\,dt$$

$$= \frac{1}{n!}\int_0^\infty e^{-(p-1)t}\frac{d^n}{dt^n}(e^{-t}t^n)\,dt$$

$$= \frac{1}{n!}\left[e^{-(p-1)t}\cdot \frac{d^{n-1}}{dt^{n-1}} e^{-t}\cdot t^n\right]_0^\infty$$

$$- \frac{1}{n!}\left[-(p-1)\int_0^\infty e^{-(p-1)t}\frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^n)dt\right]$$

$$= 0 + \frac{p-1}{n!}\int_0^\infty e^{-(p-1)t}\frac{d^{n-1}}{dt^{n-1}}(e^{-t}\cdot t^n)dt, \quad p>1$$

[$\because$ each term in $\frac{d^{n-1}}{dt^{n-1}}(e^{-t}\cdot t^n)$ contains some integral power of $t$].

Proceeding similarly again and again, we have

$$L\{L_n(t)\} = \frac{(p-1)^n}{n!}\int_0^\infty e^{-(p-1)t}(e^{-t}t^n)dt$$

$$= \frac{(p-1)^n}{n!}\int_0^\infty e^{-pt}\,t^n dt$$

$$= \frac{(p-1)^n}{n!}L\{t^n\}$$

$$= \frac{(p-1)^n}{n!}\cdot\frac{n!}{p^{n+1}}$$

$$= \frac{(p-1)^n}{p^{n+1}}$$

→ show that $L\{(1+te^{-t})^3\} = \frac{1}{p} + \frac{3}{(p+1)^2} + \frac{6}{(p+2)^3} + \frac{6}{(p+3)^4}$

Sol'n: we have

$$L\{(1+te^{-t})^3\} = L\{1+3te^{-t}+3t^2e^{-2t} + t^3e^{-3t}\}$$

$$= \frac{1}{p} + 3\frac{1!}{(p+1)^2} + 3\frac{2!}{(p+2)^3} + \frac{3!}{(p+3)^4}$$

$$= \frac{1}{p} + \frac{3}{(p+1)^2} + \frac{6}{(p+2)^3} + \frac{6}{(p+3)^4}$$

Aliter

$$L\{(1+te^{-t})^3\} = \int_0^\infty (1+te^{-t})^3\cdot e^{-pt}dt$$

$$= \int_0^\infty [e^{-pt} + 3te^{-(p+1)t} + 3t^2e^{-(p+2)t} + t^3e^{-(p+3)t}]dt$$

$$= \int_0^\infty e^{-pt}dt + 3\int_0^\infty t e^{-(p+1)t}dt + 3\int_0^\infty t^2 e^{-(p+2)t}dt + \int_0^\infty t^3 e^{-(p+3)t}dt$$

$$= \frac{1}{p} + \frac{3}{(p+1)^2} + \frac{6}{(p+2)^3} + \frac{6}{(p+3)^4}, \quad p>0$$

[$\because \int_0^\infty e^{-at}t^{n-1}dt = \frac{\Gamma(n)}{a^n}$, if $a>0$ and $n>0$.]

## * The Inverse Laplace Transform

**Definition:** If $f(P)$ is the Laplace transform of a function $F(t)$, i.e, $L\{F(t)\} = f(P)$, then $F(t)$ is called the Inverse Laplace transform of $f(P)$ and is written as

$F(t) = L^{-1}\{f(P)\}$.

**Ex:** $L\{t^n\} = \frac{n!}{p^{n+1}}$

$$\therefore t^n = L^{-1}\left\{\frac{n!}{p^{n+1}}\right\}$$

**Linearity Properties:**

Let $f_1(P)$ and $f_2(P)$ be the Laplace transforms of functions $F_1(t)$ and $F_2(t)$ respectively and $C_1, C_2$ be two constants, then

$$L^{-1}\{C_1 f_1(P) + C_2 f_2(P)\}$$

$$= C_1 L^{-1}\{f_1(P)\} + C_2 L^{-1}\{f_2(P)\}$$

$$= C_1 F_1(t) + C_2 F_2(t)$$

**Proof:** Given $L\{F_1(t)\} = f_1(P)$
$\Rightarrow F_1(t) = L^{-1}\{f_1(P)\}$

and $L\{F_2(t)\} = f_2(P) \Rightarrow F_2(t) = L^{-1}\{f_2(P)\}$

we have

$$L\{C_1F_1(t) + C_2F_2(t)\} = C_1 L\{F_1(t)\} + C_2 L\{F_2(t)\}$$

$$= C_1 f_1(P) + C_2 f_2(P)$$

$$\therefore L^{-1}\{C_1 f_1(P) + C_2 f_2(P)\} = C_1F_1(t) + C_2F_2(t)$$

$$= C_1 L^{-1}\{f_1(P)\} + C_2 L^{-1}\{f_2(P)\}$$

→ Find (i) $L^{-1}\left\{\frac{1}{P}\right\}$, $P>0$

**Sol$^n$:** Since $L\{1\} = \frac{1}{P}$

$$\therefore L^{-1}\left\{\frac{1}{P}\right\} = 1$$

→ (ii) $L^{-1}\left\{\frac{1}{p^{n+1}}\right\}$, $n$ is any real number such that $n>-1$

**Sol$^n$:** Since $L\{t^n\} = \frac{\Gamma(n+1)}{p^{n+1}}$, $P>0, n>-1$

$$\therefore L^{-1}\left\{\frac{\Gamma(n+1)}{p^{n+1}}\right\} = t^n$$

$$\Rightarrow L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}, \quad n>-1, P>0$$

If $n$ is +ve integer, then $\Gamma(n+1) = n!$

$\therefore$ from ① $L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{n!}$, $P>0$.

→ (iii) $L^{-1}\left\{\frac{1}{P-a}\right\}$

**Sol$^n$:** Since $L\{e^{at}\} = \frac{1}{P-a}$

$$\therefore L^{-1}\left\{\frac{1}{P-a}\right\} = e^{at}$$

→ Find (i) $L^{-1}\left\{\frac{1}{P^2+a^2}\right\}_{P>0}$ (ii) $L^{-1}\left\{\frac{P}{P^2+a^2}\right\}_{P>0}$

(iii) $L^{-1}\left\{\frac{1}{P^2-a^2}\right\}_{P>|a|}$ (iv) $L^{-1}\left\{\frac{P}{P^2-a^2}\right\}_{P>|a|}$

Sol'n: (i) Since $L\{\sin at\} = \frac{a}{p^2+a^2}$

$\therefore L^{-1}\left\{\frac{a}{p^2+a^2}\right\} = \sin at$

$\Rightarrow L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{1}{a}\sin at$

(ii) Since $L\{\cos at\} = \frac{p}{p^2+a^2}$

$\therefore L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$

(iii) Since $L\{\sinh at\} = \frac{a}{p^2-a^2}$

$\therefore L^{-1}\left\{\frac{a}{p^2-a^2}\right\} = \sinh at$

$\Rightarrow L^{-1}\left\{\frac{1}{p^2-a^2}\right\} = \frac{1}{a}\sinh at$

(iv) Since $L\{\cosh at\} = \frac{p}{p^2-a^2}$

$\therefore L^{-1}\left\{\frac{p}{p^2-a^2}\right\} = \cosh at$

→ Find (i) $L^{-1}\left\{\frac{1}{p^4}\right\}$ (ii) $L^{-1}\left\{\frac{1}{p^2+4}\right\}$

(iii) $L^{-1}\left\{\frac{4}{(p^2-2)}\right\}$ (iv) $L^{-1}\left\{\frac{1}{\sqrt{p}}\right\}$

Sol'n: (i) $L^{-1}\left\{\frac{1}{p^4}\right\} = L^{-1}\left\{\frac{1}{p^{3+1}}\right\} = \frac{t^3}{3!}$

$\left[\because L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{n!}\right.$ if $n$ is +ve integer$\left.\right]$

(iv) $L^{-1}\left\{\frac{1}{\sqrt{p}}\right\} = L^{-1}\left\{\frac{1}{p^{1/2}}\right\}$

$= L^{-1}\left\{\frac{1}{p^{-1/2+1}}\right\}$

$= \frac{t^{-1/2}}{\Gamma_{-1/2+1}}$ $(n = -1/2)$

$= \frac{t^{-1/2}}{\Gamma_{1/2}}$ $\left[\because L\{t^n\} = \frac{\Gamma_{n+1}}{p^{n+1}}, n>-1\right.$

$\left.\Rightarrow L^{-1}\left(\frac{1}{p^{n+1}}\right) = \frac{t^n}{\Gamma_{n+1}}\right]$

$= \frac{t^{1/2}}{\sqrt{\pi}}$ $(\because \Gamma_{1/2} = \sqrt{\pi})$

→ Find (i) $L^{-1}\left\{\frac{1}{p^{7/2}}\right\}$

(ii) $L^{-1}\left\{\frac{p}{p^2+2} + \frac{6p}{p^2-16} + \frac{3}{p-3}\right\}$

Sol'n:
(i) $L^{-1}\left\{\frac{1}{p^{7/2}}\right\} = L^{-1}\left\{\frac{1}{p^{5/2+1}}\right\}$

$= \frac{t^{5/2}}{\Gamma_{5/2+1}} = \frac{t^{5/2}}{\Gamma_{7/2}}$

$= \frac{t^{5/2}}{\frac{5}{2}\cdot\frac{3}{2}\frac{1}{2}\Gamma_{1/2}}$

$= \frac{t^{5/2}}{\frac{15}{8}\sqrt{\pi}}$

$= \frac{t^{1/2}t^2}{\frac{15}{8}\sqrt{\pi}}$

$= \frac{8t^2}{15}\sqrt{\frac{t}{\pi}}$

(ii) $L^{-1}\left\{\frac{p}{p^2+2} + \frac{6p}{p^2-16} + \frac{3}{p-3}\right\}$

$= L^{-1}\left\{\frac{p}{p^2+(\sqrt{2})^2}\right\} + 6L^{-1}\left\{\frac{p}{p^2-4^2}\right\} + 3L^{-1}\left\{\frac{1}{p-3}\right\}$

$= \cos\sqrt{2}t + 6\cosh 4t + 3e^{3t}$.

→ Find (i) $L^{-1}\left\{\frac{6}{2p-3} - \frac{3+4p}{9p^2-16} + \frac{8-6p}{16p^2+9}\right\}$

(ii) $L^{-1}\left\{\frac{3}{p^2-3} + \frac{3p+2}{p^3} - \frac{3p-27}{p^2+9} + \frac{6-30\sqrt{p}}{p^4}\right\}$

Sol'n: (i) $L^{-1}\left\{\frac{6}{2p-3} - \frac{3+4p}{9p^2-16} + \frac{8-6p}{16p^2+9}\right\}$

$= L^{-1}\left\{\frac{6}{2[p-\frac{3}{2}]}\right\} - L^{-1}\left\{\frac{3}{9p^2-16}\right\} + 4L^{-1}\left\{\frac{p}{9p^2-16}\right\}$

$+ 8L^{-1}\left\{\frac{1}{16p^2+9}\right\} - 6L^{-1}\left\{\frac{p}{16p^2+9}\right\}$

$= 3L^{-1}\left\{\frac{1}{p-3/2}\right\} - \frac{3}{9}L^{-1}\left\{\frac{1}{p^2-(4/3)^2}\right\} +$

$\frac{4}{9}L^{-1}\left\{\frac{p}{p^2-(4/3)^2}\right\} + \frac{8}{16}L^{-1}\left\{\frac{1}{p^2+(3/4)^2}\right\} - \frac{6}{16}L^{-1}\left\{\frac{p}{p^2+(3/4)^2}\right\}$

$= 3e^{3/2\,t} - \frac{1}{3}\,\frac{1}{4/3}\sinh\frac{4}{3}t + \frac{4}{9}\cdot\cosh\frac{4}{3}t +$

$\frac{1}{2}\cdot\frac{1}{(3/4)}\sin 3/4\,t - \frac{3}{8}\cos 3/4\,t$

$= 3e^{3/2\,t} - \frac{1}{4}\sinh 4/3\,t + 4/9\cosh\frac{4t}{3}$

$+ \frac{2}{3}\sin 3/4\,t - 3/8\cos 3/4\,t.$

→ Prove that

$L^{-1}\left\{\frac{5}{p^2} + \left(\frac{\sqrt{p}-1}{p}\right)^2 - \frac{7}{3p+2}\right\}$

$= 1 + 6t - 4\sqrt{\frac{t}{\pi}} - \frac{7}{3}e^{-2/3\,t}$

→ show that

(i) $L^{-1}\left\{\frac{1}{p}\cos\frac{1}{p}\right\} = 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \cdots$

(ii) $L^{-1}\left\{\frac{1}{p}\sin\frac{1}{p}\right\} = t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \frac{t^7}{(7!)^2} + \cdots$

(iii) $L^{-1}\left\{\frac{1}{p^3+1}\right\} = \frac{t^2}{2!} - \frac{t^5}{5!} + \frac{t^8}{8!} - \frac{t^{11}}{11!} + \cdots$

$\left(\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$

Sol'n: (i) $L^{-1}\left\{\frac{1}{p}\cos\frac{1}{p}\right\}$

$= L^{-1}\left\{\frac{1}{p}\left[1 - \frac{(1/p)^2}{2!} + \frac{(1/p)^4}{4!} - \frac{(1/p)^6}{6!} \cdots\right]\right\}$

$= L^{-1}\left\{\frac{1}{p} - \frac{1}{p^3\,2!} + \frac{1}{4!\,p^5} - \frac{1}{6!\,p^7} \cdots\right\}$

$= L^{-1}\left\{\frac{1}{p}\right\} - \frac{1}{2!}L^{-1}\left\{\frac{1}{p^3}\right\} + \frac{1}{4!}L^{-1}\left\{\frac{1}{p^5}\right\} \cdots$

$- \frac{1}{6!}L^{-1}\left\{\frac{1}{p^7}\right\} + \cdots$

$= 1 - \frac{1}{2!}\frac{t^2}{2!} + \frac{1}{4!}\frac{t^4}{4!} - \frac{1}{6!}\frac{t^6}{6!} + \cdots$

$= 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \cdots$

(iii) $L^{-1}\left\{\frac{1}{p^3+1}\right\} = L^{-1}\left\{\frac{1}{p^3\left[1+\frac{1}{p^3}\right]}\right\}$

$\left[(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \cdots\right]$

$= L^{-1}\left\{\frac{1}{p^3}\left(1+\frac{1}{p^3}\right)^{-1}\right\}$

$= L^{-1}\left\{\frac{1}{p^3}\left[1 - \frac{1}{p^3} + \left(\frac{1}{p^3}\right)^2 - \left(\frac{1}{p^3}\right)^3 + \left(\frac{1}{p^3}\right)^4 - \cdots\right]\right\}$

$= L^{-1}\left\{\frac{1}{p^3}\right\} - L^{-1}\left\{\frac{1}{p^6}\right\} + L^{-1}\left\{\frac{1}{p^9}\right\} - L^{-1}\left\{\frac{1}{p^{12}}\right\} + \cdots$

$= \frac{t^2}{2!} - \frac{t^5}{5!} + \frac{t^8}{8!} - \frac{t^{11}}{11!} + \cdots$

→ show that $L^{-1}\left\{\frac{e^{-1/p}}{p}\right\} = J_0\,2\sqrt{t}$

where

$$\boxed{J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} \cdots}$$

→ If $L^{-1}\left\{\frac{p^2-1}{(p^2+1)^2}\right\} = t\cos t$, Prove that

$$L^{-1}\left\{\frac{9p^2-1}{(9p^2+1)^2}\right\} = \frac{1}{9}t\cos t/_3$$

Sol$^n$: Given $L^{-1}\left\{\frac{p^2-1}{(p^2+1)^2}\right\} = t\cos t$

then $L^{-1}\left\{\frac{9p^2-1}{(9p^2+1)^2}\right\} = L^{-1}\left\{\frac{(3p)^2-1}{((3p)^2+1)^2}\right\}$

$$\left[\because L^{-1}(f(ap)) = \frac{1}{a}F(t/a)\right]$$

$$= \frac{1}{3}\, t/_3 \cos t/_3$$

$$= \frac{t}{9}\cos t/_3$$

→ If $L^{-1}\left\{\frac{p}{p^2-16}\right\} = \cosh 4t$, then Prove

$$L^{-1}\left\{\frac{p}{2p^2-8}\right\} = \frac{1}{2}\cosh 2t$$

→ Find $L^{-1}\left\{\frac{e^{-5p}}{(p-2)^4}\right\}$

It is in the form of $L^{-1}\{e^{-ap}f(p)\}$

Sol$^n$: Let $f(p) = \frac{1}{(p-2)^4}$

i.e. $L\{F(t)\} = \frac{1}{(p-2)^4}$

$$\Rightarrow F(t) = L^{-1}\left\{\frac{1}{(p-2)^4}\right\}$$

$$= e^{2t}L^{-1}\left\{\frac{1}{p^4}\right\}$$

$$= e^{2t}\frac{t^3}{3!} = \frac{1}{6}t^3e^{2t}$$

Hence by second shifting theorem, we have

$$L^{-1}\{e^{-ap}f(p)\} = \begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$$

$$\therefore L^{-1}\{e^{-5p}f(p)\} = \begin{cases} F(t-5), & t>a \\ 0, & t<a \end{cases}$$

$$\therefore L^{-1}\left\{e^{-5p}\frac{1}{(p-2)^4}\right\} = \begin{cases} \frac{1}{6}(t-5)^3e^{2(t-5)}, & t>5 \\ 0, & t<5 \end{cases}$$

$$= \frac{1}{6}(t-5)^3e^{2(t-5)}$$

interms of Heaviside unit stepfunction

→ Find $L^{-1}\left\{\frac{e^{4-3p}}{(p+4)^{5/2}}\right\}$

Sol$^n$: Let $f(p) = \frac{1}{(p+4)^{5/2}}$

$$F(t) = L^{-1}\left\{\frac{1}{(p+4)^{5/2}}\right\} = e^{-4t}L^{-1}\left\{\frac{1}{p^{5/2}}\right\}$$

$$= e^{-4t}\frac{t^{5/2-1}}{\Gamma 5/2} \qquad \left| \; L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{\Gamma n+1},\; n>-1 \right.$$

$$= e^{-4t}\frac{t^{3/2}}{3/2\cdot\frac{1}{2}\Gamma\frac{1}{2}}$$

$$= 4/_3\, e^{-4t}\frac{t^{3/2}}{\sqrt{\pi}}$$

$$\therefore L^{-1}\left\{\frac{e^{4-3p}}{(p+4)^{5/2}}\right\} = e^4L^{-1}\left\{\frac{e^{-3p}}{(p+4)^{5/2}}\right\}$$

∴ by second shifting theorem, we have

$$L^{-1}\{e^{-ap}f(p)\} = \begin{cases} F(t-a) & t>a \\ 0 & t<a \end{cases}$$

$$\text{i.e, } e^4L^{-1}\{e^{-3p}f(p)\} = \begin{cases} e^4F(t-3) & t>a \\ 0 & t<a \end{cases}$$

Hence by the definition of inverse L.T we get

$$L^{-1}\{e^{-ap} f(p)\} = G(t)$$

Note: The result of this theorem can also be expressed in the following two ways

1. $L^{-1}\{f(p)\} = F(t)$, $L^{-1}\{e^{-pa} f(p)\} = \begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$

2. $L^{-1}\{f(p)\} = F(t)$, $L^{-1}\{e^{-pa} f(p)\} = F(t-a)H(t-a)$

where $H(t-a)$ is Heaviside unit step function which is defined as follows:

$$H(t-a) = \begin{cases} 1, & \text{when } t>a \\ 0, & \text{when } t<a. \end{cases}$$

## * Change of Scale Property:

Theorem: If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{f(ap)\} = \frac{1}{a} F(t/a)$$

Sol'n: Given $L^{-1}\{f(p)\} = F(t)$

$$\Rightarrow f(p) = L\{F(t)\}$$

$$\therefore f(p) = \int_0^\infty e^{-pt} F(t)\,dt$$

$$\therefore f(ap) = \int_0^\infty e^{-apt} F(t)\,dt$$

Put $at = x \Rightarrow dt = \frac{dx}{a}$

$t = x/a$

$$= \frac{1}{a}\int_0^\infty e^{-px} F\left(\frac{x}{a}\right) dx$$

$$= \frac{1}{a}\int_0^\infty e^{-pt} F(t/a)\,dt$$

$$\left(\because \int_a^b f(x)\,dx = \int_a^b F(t)\,dt\right)$$

$$= \frac{1}{a} L\{F(t/a)\}$$

$$= L\left\{\frac{1}{a} F(t/a)\right\}$$

$$\therefore f(ap) = L\left\{\frac{1}{a} F(t/a)\right\}$$

$$\Rightarrow L^{-1}\{f(ap)\} = \frac{1}{a} F(t/a)$$

→ Find $L^{-1}\left\{\frac{1}{p^2-6p+10}\right\}$

Sol'n: $L^{-1}\left\{\frac{1}{p^2-6p+10}\right\} = L^{-1}\left\{\frac{1}{(p-3)^2+1}\right\}$

$$= e^{3t} L^{-1}\left\{\frac{1}{p^2+1}\right\}$$

$$= e^{3t}\sin t \quad \text{(using first shifting theorem)}$$

→ Find (i) $L^{-1}\left\{\frac{1}{p^2+8p+16}\right\}$.

(ii) $L^{-1}\left\{\frac{p-1}{(p+3)(p^2+2p+2)}\right\}$

Sol'n (i) $L^{-1}\left\{\frac{1}{p^2+8p+16}\right\} = L^{-1}\left\{\frac{1}{(p+4)^2}\right\}$

$$= e^{-4t}\left\{\frac{1}{p^2}\right\} \quad \text{(by using first shifting theorem)}$$

$$= e^{-4t}\frac{t}{1!} = te^{-4t}.$$

i) $L^{-1}\left\{\dfrac{P-1}{(P+3)(P^2+2P+2)}\right\}$

Now $\dfrac{P-1}{(P+3)(P^2+2P+2)} = \dfrac{A}{P+3} + \dfrac{BP+C}{P^2+2P+2}$

$P-1 = A(P^2+2P+2)+(BP+C)(P+3)$

$P-1 = (A+B)P^2+(2A+3B+C)P+2A+3C$

Comparing on bothsides

$A+B=0 \Rightarrow A=-B$

$2A+3B+C=1 \Rightarrow 2(-B)+3B+C=1$

$\Rightarrow B+C=1$

$2A+3C=-1$

$\Rightarrow 2(-B)+3C=-1$

$\Rightarrow -2B+3C=-1$

$2B+2C=2$

$-2B+3C=-1$

$5C=1$

$\boxed{C=1/5}$

$B+C=1$

$B=1-1/5$

$\boxed{B=4/5}$

$A+B=0$

$\boxed{A=-4/5}$

$\therefore \dfrac{P-1}{(P+3)(P^2+2P+2)} = -\dfrac{4}{5(P+3)} + \dfrac{4/5\,P+1/5}{P^2+2P+2}$

$= \dfrac{-4}{5(P+3)} + \dfrac{4P+1}{5(P^2+2P+2)}$

$L^{-1}\left\{\dfrac{P-1}{(P+3)(P^2+2P+2)}\right\}$

$= \dfrac{-4}{5}L^{-1}\left\{\dfrac{1}{P+3}\right\} + \dfrac{1}{5}L^{-1}\left\{\dfrac{4P+1}{(P^2+2P+2)}\right\}$

$= \dfrac{-4}{5}e^{-3t}L^{-1}\left\{\dfrac{1}{P}\right\} + \dfrac{1}{5}L^{-1}\left\{\dfrac{4(P+1)-3}{(P+1)^2+1}\right\}$

$= -\dfrac{4}{5}e^{-3t} + \dfrac{e^{-t}}{5}L^{-1}\left\{\dfrac{4P-3}{P^2+1}\right\} \quad \left(\because L^{-1}\left\{\dfrac{1}{P}\right\}=1\right)$

$= -4/5\, e^{-3t} + \dfrac{e^{-5}}{5}\left[L^{-1}\left\{\dfrac{4P}{P^2+1}\right\} + L^{-1}\left\{\dfrac{-3}{P^2+1}\right\}\right]$

$= -4/5\, e^{-3t} + \dfrac{e^{-5}}{5}(4\cos t - 3\sin t)$

$L^{-1}\left\{\dfrac{3P-2}{P^2-4P+20}\right\} \overset{?}{=} L^{-1}\left\{\dfrac{3P-2}{(P-2)^2+16}\right\}$

$= L^{-1}\left\{\dfrac{3(P-2)+4}{(P-2)^2+4^2}\right\}$

$= e^{2t}L^{-1}\left\{\dfrac{3P+4}{P^2+4^2}\right\}$

$= e^{2t}\left[L^{-1}\left\{\dfrac{3P}{P^2+4^2}\right\} + L^{-1}\left\{\dfrac{4}{P^2+4^2}\right\}\right]$

$= e^{2t}\left[3\cos 2t + 4\sin 4t\right]$

$L^{-1}\left\{\dfrac{3P+7}{P^2-2P-3}\right\} \overset{?}{=} L^{-1}\left\{\dfrac{3P+7}{(P-1)^2-4^2}\right\}$

$= L^{-1}\left\{\dfrac{3(P-1)+10}{(P-1)^2-4}\right\}$

$= e^t L^{-1}\left\{\dfrac{3P+10}{P^2-2^2}\right\}$

$= e^t\left[3L^{-1}\left\{\dfrac{P}{P^2-2^2}\right\} + 10L^{-1}\left\{\dfrac{1}{P^2-2^2}\right\}\right]$

$= e^t\left[3\cosh 2t + \dfrac{10}{2}\sinh 2t\right]$

$= e^t\left[3\cosh 2t + 5\sinh 2t\right]$

$\boxed{\because L^{-1}\left\{\dfrac{1}{P^2-a^2}\right\} = \dfrac{1}{a}\sinh at}$

→ Find (i) $L^{-1}\left\{\frac{1}{(p+a)^n}\right\}$ (ii) $L^{-1}\left\{\frac{p}{(p+1)^{5/2}}\right\}$

(iii) $L^{-1}\left\{\frac{p}{(p+1)^5}\right\}$ (iv) $L^{-1}\left\{\frac{3p+2}{4p^2+12p+9}\right\}$

→ Evaluate (i) $L^{-1}\left\{\frac{1}{\sqrt{2p+3}}\right\}$ (ii) $L^{-1}\left\{\frac{1}{(8p-27)^{1/3}}\right\}$

Sol'n (ii): $L^{-1}\left\{\frac{1}{(8p-27)^{1/3}}\right\} = L^{-1}\left\{\frac{1}{8^{1/3}\left(p-\frac{27}{8}\right)^{1/3}}\right\}$

$$\Rightarrow \frac{1}{2} L^{-1}\left\{\frac{1}{\left(p-\frac{27}{8}\right)^{1/3}}\right\}$$

$$= \frac{1}{2} e^{27/8\, t} L^{-1}\left\{\frac{1}{p^{1/3}}\right\}$$

$$= \frac{e^{27/8 t}}{2} \cdot \frac{t^{1/3-1}}{\Gamma 1/3}$$

$\boxed{L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{\Gamma n+1}}$

$$= \frac{e^{27/8 t}}{2} \cdot \frac{t^{-2/3}}{\Gamma 1/3}$$

→ If $L^{-1}\left\{\frac{e^{-1/p}}{p^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$, find

$L^{-1}\left\{\frac{e^{-a/p}}{p^{1/2}}\right\}$ where $a>0$.

Sol'n: Since $L^{-1}\{f(ap)\} = \frac{1}{a} F(t/a)$

Hence $L^{-1}\left\{\frac{e^{-1/p}}{\sqrt{p}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$ gives

$$L^{-1}\left\{\frac{e^{-1/pk}}{\sqrt{pk}}\right\} = \frac{1}{k} \frac{\cos 2\sqrt{t/k}}{\sqrt{\frac{\pi t}{k}}}$$

$$\Rightarrow \frac{1}{\sqrt{k}} L^{-1}\left\{\frac{e^{-1/pk}}{\sqrt{p}}\right\} = \frac{1}{k} \frac{\cos\left(2\sqrt{t/k}\right)}{\sqrt{\frac{\pi t}{k}}}$$

$$\Rightarrow L^{-1}\left\{\frac{e^{-1/pk}}{\sqrt{p}}\right\} = \frac{\sqrt{k}}{k} \frac{\cos 2\left(t/k\right)^{1/2}}{\frac{\sqrt{t}}{\sqrt{k}}\sqrt{\pi t}}$$

Putting $k = 1/a$

$$\therefore L^{-1}\left\{\frac{e^{-a/p}}{\sqrt{p}}\right\} = \frac{\cos 2(at)^{1/2}}{\sqrt{\pi t}}$$

$$= \frac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$$

Imp → $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{1}{2} t \sin t$, find

$L^{-1}\left\{\frac{32p}{(16p^2+1)^2}\right\}$

Sol'n: Given

$L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{1}{2} t \sin t$

Since $f(ap) = \frac{1}{a} F(t/a)$

$$\therefore L^{-1}\left\{\frac{ap}{((ap)^2+1)^2}\right\} = \frac{1}{2}\left(\frac{1}{a}\right) \frac{t}{a} \sin t/a$$

Putting $a = 4$

$$L^{-1}\left\{\frac{4p}{(16p^2+1)^2}\right\} = \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{t}{4} \sin t/4$$

$$L^{-1}\left\{\frac{4p}{(16p^2+1)^2}\right\} = \frac{1}{8} \frac{t}{4} \sin t/4$$

$$\Rightarrow 8L^{-1}\left\{\frac{4p}{(16p^2+1)^2}\right\} = \frac{t}{4} \sin t/4.$$

$$\Rightarrow L^{-1}\left\{\frac{32p}{(16p^2+1)^2}\right\} = t/4 \sin t/4$$

→ If $L^{-1}\left\{\frac{p^2-1}{(p^2+1)^2}\right\} = t\cos t$, Prove that

$$L^{-1}\left\{\frac{9p^2-1}{(9p^2+1)^2}\right\} = \frac{1}{9}t\cos t/3$$

Sol'n: Given $L^{-1}\left\{\frac{p^2-1}{(p^2+1)^2}\right\} = t\cos t$

then $L^{-1}\left\{\frac{9p^2-1}{(9p^2+1)^2}\right\} = L^{-1}\left\{\frac{(3p)^2-1}{((3p)^2+1)^2}\right\}$

$\left[\because L^{-1}(f(ap)) = \frac{1}{a}F(t/a)\right]$

$$= \frac{1}{3}\,\frac{t}{3}\cos\frac{t}{3}$$

$$= \frac{t}{9}\cos t/3$$

→ If $L^{-1}\left\{\frac{p}{p^2-16}\right\} = \cosh 4t$, then Prove

$$L^{-1}\left\{\frac{p}{2p^2-8}\right\} = \frac{1}{2}\cosh 2t$$

→ Find $L^{-1}\left\{\frac{e^{-5p}}{(p-2)^4}\right\}$

It is in the form of $L^{-1}\{e^{-ap}f(p)\}$

Sol'n: Let $f(p) = \frac{1}{(p-2)^4}$

i.e. $L\{F(t)\} = \frac{1}{(p-2)^4}$

$$\Rightarrow F(t) = L^{-1}\left\{\frac{1}{(p-2)^4}\right\}$$

$$= e^{2t}L^{-1}\left\{\frac{1}{p^4}\right\}$$

$$= e^{2t}\frac{t^3}{3!} = \frac{1}{6}t^3e^{2t}$$

Hence by second shifting theorem, we have

$$L^{-1}\{e^{-ap}f(p)\} = \begin{cases} F(t-a); & t>a \\ 0, & t<a \end{cases}$$

$$\therefore L^{-1}\{e^{-5p}f(p)\} = \begin{cases} F(t-5), & t>a \\ 0, & t<a \end{cases}$$

$$\therefore L^{-1}\left\{e^{-5p}\frac{1}{(p-2)^4}\right\} = \begin{cases} \frac{1}{6}(t-5)^3e^{2(t-5)}, & t>5 \\ 0, & t<5 \end{cases}$$

$$= \frac{1}{6}(t-5)^3e^{2(t-5)}$$

interms of Heaviside unit step function

→ Find $L^{-1}\left\{\frac{e^{4-3p}}{(p+4)^{5/2}}\right\}$

Sol'n: Let $f(p) = \frac{1}{(p+4)^{5/2}}$

$$F(t) = L^{-1}\left\{\frac{1}{(p+4)^{5/2}}\right\} = e^{-4t}L^{-1}\left\{\frac{1}{p^{5/2}}\right\}$$

$$= e^{-4t}\frac{t^{5/2-1}}{\Gamma 5/2}$$

$\left| L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{\Gamma n+1},\ n>-1 \right.$

$$= e^{-4t}\frac{t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}$$

$$= \frac{4}{3}e^{-4t}\frac{t^{3/2}}{\sqrt{\pi}}$$

$$\therefore L^{-1}\left\{\frac{e^{4-3p}}{(p+4)^{5/2}}\right\} = e^4L^{-1}\left\{\frac{e^{-3p}}{(p+4)^{5/2}}\right\}$$

∴ by second shifting theorem, we have

$$L^{-1}\{e^{-ap}f(p)\} = \begin{cases} F(t-a) & t>a \\ 0 & t<a \end{cases}$$

i.e, $e^4L^{-1}\{e^{-3p}f(p)\} = \begin{cases} e^4F(t-3) & t>a \\ 0 & t<a \end{cases}$

$$e^{4} L^{-1}\left\{\frac{e^{-3p}}{(p+4)^{3/2}}\right\}$$

$$= \begin{cases} e^{4}\frac{4}{3\sqrt{\pi}} e^{-4(t-3)} (t-3)^{3/2} ; & t>3 \\ 0 & , t<3 \end{cases}$$

$$= \begin{cases} \frac{4}{3\sqrt{\pi}} e^{-4(t-4)} (t-3)^{3/2}, & t>3 \\ 0 & , t<3 \end{cases}$$

$$= \frac{4}{3\sqrt{\pi}} e^{-4(t-4)} (t-3)^{3/2} H(t-3)$$

interms of Heaviside unit step function.

→ For $a>0$, Prove that $L^{-1}\{f(p)\}=F(t)$ implies $L^{-1}\{f(ap+b)\}=\frac{1}{a}e^{-bt/a}F(t/a)$

Sol'n: we have $L^{-1}\{f(p)\}=F(t)$

$$\Rightarrow f(p) = L\{F(t)\} = \int_0^\infty e^{-pt}F(t)dt$$

$$\Rightarrow f(ap+b) = \int_0^\infty e^{-(ap+b)t} F(t)\,dt$$

$$= \int_0^\infty e^{-apt}e^{-bt}F(t)\,dt$$

$$= \int_0^\infty e^{-p\tau}e^{-b\tau/a}F(\tau/a)\frac{d\tau}{a}$$

Putting $at=\tau$ $\Rightarrow t=\tau/a$ $\Rightarrow dt=\frac{1}{a}d\tau$

$$= \int_0^\infty e^{-pt}\left\{\frac{1}{a}e^{-bt/a}F(t/a)\right\}dt$$

by property of definite integrals $\int_a^b f(x)\,dx = \int_a^b f(t)dt$.

$$= L\left\{\frac{1}{a}e^{-bt/a}F(t/a)\right\}$$

$$\therefore f(ap+b) = L\left\{\frac{1}{a}e^{-bt/a}F(t/a)\right\}$$

$$\Rightarrow L^{-1}\{f(ap+b)\} = \frac{1}{a}e^{-bt/a}F(t/a)$$

→ Evaluate $L^{-1}\left\{\frac{p+1}{p^2+6p+25}\right\}$

Sol'n: $L^{-1}\left\{\frac{p+1}{p^2+6p+25}\right\} = L^{-1}\left\{\frac{p+1}{(p+3)^2+16}\right\}$

$$= L^{-1}\left\{\frac{(p+3)-2}{(p+3)^2+4^2}\right\}$$

$$= e^{-3t}L^{-1}\left\{\frac{p-2}{p^2+4^2}\right\}$$

$$= e^{-3t}\left[L^{-1}\left\{\frac{p}{p^2+4^2}\right\} - L^{-1}\left\{\frac{2}{p^2+4^2}\right\}\right]$$

$$= e^{-3t}\left[\cos 2t - 2\cdot\frac{1}{4}\sin 4t\right]$$

$$= e^{-3t}\left[\cos 2t - \tfrac{1}{2}\sin 4t\right]$$

→ Evaluate $L^{-1}\left\{\frac{6p^2+22p+18}{p^3+6p^2+11p+6}\right\}$

Sol'n:

$$L^{-1}\left\{\frac{6p^2+22p+18}{p^3+6p^2+11p+6}\right\} = L^{-1}\left\{\frac{6p^2+22p+18}{(p+1)(p+2)(p+3)}\right\}$$

$$= L^{-1}\left\{\frac{1}{p+1}+\frac{2}{p+2}+\frac{3}{p+3}\right\}$$

$$= L^{-1}\left\{\frac{1}{p+1}\right\} + L^{-1}\left\{\frac{2}{p+2}\right\} + L^{-1}\left\{\frac{3}{p+3}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{1}{p}\right\} + 2e^{-2t}L^{-1}\left\{\frac{1}{p}\right\} + 3e^{-3t}L^{-1}\left\{\frac{1}{p}\right\}$$

$$= e^{-t}+2e^{-2t}+3e^{-3t}$$

→ Evaluate $L^{-1}\left\{\frac{3(p^2+2p+3)}{(p^2+2p+2)(p^2+2p+5)}\right\}$

Sol$^n$.

$$L^{-1}\left\{\frac{3(p^2+2p+3)}{(p^2+2p+2)(p^2+2p+5)}\right\}$$

$$=L^{-1}\left\{\frac{1}{p^2+2p+2}+\frac{2}{p^2+2p+5}\right\}$$

$$=L^{-1}\left\{\frac{1}{(p+1)^2+1}\right\}+2L^{-1}\left\{\frac{1}{(p+1)^2+4}\right\}$$

$$=e^{-t}\sin t+2e^{-t}\sin 2t$$

→ Prove that $L^{-1}\left\{\frac{4p+5}{(p-1)^2(p+2)}\right\}$

$$=3te^t+\tfrac{1}{3}e^t-\tfrac{1}{3}e^{-2t}$$

→ Evaluate $L^{-1}\left\{\frac{4p+5}{(p-4)^2(p+3)}\right\}$

→ Find $L^{-1}\left\{\frac{5p^2-15p-11}{(p+1)(p-2)^3}\right\}$

Sol$^n$: $\frac{5p^2-15p-11}{(p+1)(p-2)^3}$

$$=\frac{-1}{3(p+1)}+\frac{1}{3(p-2)}+\frac{4}{(p-2)^2}-\frac{7}{(p-2)^3}$$

→ Prove that

$$L^{-1}\left\{\frac{2p+1}{(p+2)^2(p-1)}\right\}=\frac{1}{3}t\left[e^t-e^{-2t}\right]$$

→ Find $L^{-1}\left\{\frac{1}{(p+1)(p^2+1)}\right\}$

→ Prove that

$$L^{-1}\left\{\frac{p}{(p^2-2p+2)(p^2+2p+2)}\right\}=\tfrac{1}{2}\sin t\cdot\sinh t$$

→ Prove that

$$L^{-1}\left\{\frac{p}{p^4+p^2+1}\right\}=\frac{2}{\sqrt{3}}\sinh\tfrac{t}{2}\sin\tfrac{1}{2}\sqrt{3}t$$

$$\left[\frac{p}{p^4+p^2+1}=\frac{p}{(p^2+1)^2-p^2}=\frac{p}{(p^2+1+p)(p^2+1-p)}\right]$$

→ Find $L^{-1}\left\{\frac{2p^3+2p^2+4p+1}{(p^2+1)(p^2+p+1)}\right\}$

→ Show that

$$L^{-1}\left\{\frac{p^2}{p^4+4a^4}\right\}=\frac{1}{2a}\left[\cosh at\ \sin at+\sinh at\ \cos at\right]$$

Sol$^n$: L.H.S $=L^{-1}\left\{\frac{p^2}{(p^2+2a^2)^2-4a^2p^2}\right\}$

$$=L^{-1}\left\{\frac{p^2}{(p^2+2a^2-2ap)(p^2+2a^2+2ap)}\right\}$$

$$=L^{-1}\left\{\frac{p}{4a(p^2-2ap+2a^2)}-\frac{p}{4a(p^2+2ap+2a^2)}\right\}$$

$$=\frac{1}{4a}L^{-1}\left\{\frac{(p-a)+a}{(p-a)^2+a^2}-\frac{(p+a)-a}{(p+a)^2+a^2}\right\}$$

$$=\frac{1}{4a}\left[e^{at}L^{-1}\left\{\frac{p+a}{p^2+a^2}\right\}-e^{-at}L^{-1}\left\{\frac{p-a}{p^2+a^2}\right\}\right]$$

$$=\frac{1}{4a}\left[e^{at}\left(L^{-1}\left\{\frac{p}{p^2+a^2}\right\}+L^{-1}\left\{\frac{a}{p^2+a^2}\right\}\right)\right.$$

$$\left.-e^{-at}\left(L^{-1}\left\{\frac{p}{p^2+a^2}\right\}-L^{-1}\left\{\frac{a}{p^2+a^2}\right\}\right)\right]$$

## Inverse Laplace Transform of derivatives:

**Theorem:** If $L^{-1}\{f(p)\} = F(t)$, then

$L^{-1}\{f^n(p)\} = (-1)^n t^n F(t)$. i.e,

$$L^{-1}\left\{\frac{d^n}{dp^n} f(p)\right\} = (-1)^n t^n F(t), \quad n = 1,2,3\ldots$$

**Proof:** we know that

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} f(p)$$

$$= (-1)^n f^n(p)$$

$$\therefore L^{-1}\{f^{(n)}(p)\} = (-1)^n t^n F(t)$$

$$(\text{or})\ L^{-1}\left\{\frac{d^n}{dp^n} f(p)\right\} = (-1)^n t^n F(t)$$

**Note:** The result of this theorem can also be written as

$$L^{-1}\left\{\frac{d^n}{dp^n} f(p)\right\} = L^{-1}\{f^{(n)}(p)\} = (-1)^n t^n L^{-1}\{f(p)\}$$

→ Find $L^{-1}\left\{\dfrac{p}{(p^2-a^2)^2}\right\}$

**Sol'n:**

Let $f(p) = \dfrac{1}{p^2-a^2}$

$$\Rightarrow \frac{d}{dp} f(p) = \frac{-2p}{(p^2-a^2)^2}$$

$$\Rightarrow \frac{p}{(p^2-a^2)^2} = -\frac{1}{2}\frac{d}{dp} f(p)$$

$$= -\frac{1}{2}\frac{d}{dp}\left(\frac{1}{p^2-a^2}\right)$$

$f(p) = \dfrac{1}{p^2-a^2}$

$L\{F(t)\} = f(p)$

$F(t) = L^{-1}\{f(p)\}$

$F(t) = L^{-1}\left\{\dfrac{1}{p^2-a^2}\right\}$

$= \dfrac{1}{a}\sinh at$

$$\therefore L^{-1}\left\{\frac{p}{(p^2-a^2)^2}\right\} = L^{-1}\left\{-\frac{1}{2}\frac{d}{dp}\left(\frac{1}{p^2-a^2}\right)\right\}$$

$$= -\frac{1}{2} L^{-1}\left\{\frac{d}{dp}\left(\frac{1}{p^2-a^2}\right)\right\}$$

$$= -\frac{1}{2}(-1)^1 t\ L^{-1}\left\{\frac{1}{p^2-a^2}\right\} \text{ by formula}$$

$$= \frac{1}{2} t \left(\frac{1}{a}\right)\sinh at$$

$$= \frac{t}{2a}\sinh at \qquad L^{-1}\left\{\frac{1}{p^2-a^2}\right\} = \frac{1}{a}$$

→ Find (i) $L^{-1}\left\{\dfrac{p}{(p^2-16)^2}\right\}$

(ii) $L^{-1}\left\{\dfrac{p}{(p^2+a^2)^2}\right\}$ (iii) $L^{-1}\left\{\dfrac{p}{(p^2+4)^2}\right\}$

→ Evaluate

(i) $L^{-1}\left\{\dfrac{p+1}{(p^2+2p+2)^2}\right\}$ (ii) $L^{-1}\left\{\dfrac{p+2}{(p^2+4p+5)^2}\right\}$

**Sol'n:** (i) $L^{-1}\left\{\dfrac{p+1}{(p^2+2p+2)^2}\right\}$

Let $f(p) = \dfrac{1}{p^2+2p+2} = (p^2+2p+2)^{-1}$

$$\Rightarrow \frac{d}{dp} f(p) = \frac{-(2p+2)}{(p^2+2p+2)^2} = \frac{-2(p+1)}{(p^2+2p+2)^2}$$

$$\Rightarrow \frac{p+1}{(p^2+2p+2)^2} = -\frac{1}{2}\frac{d}{dp} f(p)$$

$$= -\frac{1}{2}\frac{d}{dp}\left(\frac{1}{p^2+2p+2}\right)$$

$$\left(\because f(p) = \frac{1}{p^2+2p+2}\right)$$

$$\therefore L^{-1}\left(\frac{p+1}{p^2+2p+2}\right) = -\frac{1}{2} L^{-1}\left\{\frac{d}{dp}\left(\frac{1}{p^2+2p+2}\right)\right\}$$

$$= -\frac{1}{2}(-1)^1 t\, L^{-1}\left\{\frac{1}{p^2+2p+2}\right\}$$

$$= \int_0^\infty L(e^{-tx^2})\,dx$$

$$= \int_0^\infty \frac{1}{p+x^2}\,dx \qquad \left[\because L\{e^{-at}\} = \frac{1}{p+a}\right]$$

$$= \int_0^\infty \frac{1}{(\sqrt{p})^2+x^2}\,dx$$

$$= \left[\frac{1}{\sqrt{p}}\tan^{-1}\left(\frac{x}{\sqrt{p}}\right)\right]_0^\infty$$

$$= \left[\frac{1}{\sqrt{p}}\frac{\pi}{2} - 0\right] = \frac{1}{\sqrt{p}}\frac{\pi}{2}.$$

$$\therefore L\{F(t)\} = \frac{1}{\sqrt{p}}\left(\frac{\pi}{2}\right)$$

$$F(t) = L^{-1}\left\{\frac{1}{\sqrt{p}}\frac{\pi}{2}\right\}$$

$$= \frac{\pi}{2} L^{-1}\left(\frac{1}{\sqrt{p}}\right)$$

$$F(t) = \frac{\pi}{2}\frac{1}{\sqrt{\pi t}} = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{t}}$$

$$\therefore \int_0^\infty e^{-tx^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{t}} \quad \text{—— (2)}$$

(vi) putting $t=1$ in eqn (2)

$$\underline{\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}}.$$

→ we know that $\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}\,dx$.

The complementary error function is defined as

$$\operatorname{erfc}(t) = 1 - \operatorname{erf}(t)$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\,dx - \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}\,dx$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\,dx + \frac{2}{\sqrt{\pi}}\int_t^0 e^{-x^2}\,dx \qquad \left[\because \operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\,dx = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1 \quad \left(\because \int_0^\infty e^{-x^2}dx = 1\right) \Rightarrow \operatorname{erf}(\infty) = 1\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\int_t^0 e^{-x^2}\,dx + \int_0^\infty e^{-x^2}\,dx\right]$$

$$\boxed{\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-x^2}\,dx.}$$

$$= \frac{1}{2\sqrt{p}}\left(\frac{1}{2}\right)\frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{\Gamma\frac{1}{4}+\frac{3}{4}}$$

(pere fn

$\because \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\frac{\Gamma m\,\Gamma n}{\Gamma m+n}$ )

$2m-1 = -\frac{1}{2}$ ; $2n-1 = \frac{1}{2}$

$2m = \frac{1}{2}$ ; $2n = 3/2$

$m = \frac{1}{4}$ ; $n = 3/4$

$$= \frac{1}{4\sqrt{p}}\frac{\Gamma\frac{1}{4}\,\Gamma 3/4}{\Gamma 1}$$

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$(\because \Gamma 1 = 1)$

$$= \frac{1}{4\sqrt{p}}\,\Gamma(1/4)\,\Gamma(3/4) = \frac{1}{4\sqrt{p}}\,\Gamma\frac{1}{4}\,\Gamma 1-\frac{1}{4} \quad \left(\text{Here } p_1 = \frac{1}{4}\right)$$

$$\left(\because \Gamma(p_1)\,\Gamma(1-p_1) = \frac{\pi}{\sin p_1 \pi}, \text{ where } 0<p_1<1\right)$$

$$= \frac{1}{4\sqrt{p}}\,\frac{\pi}{\sin\left(\frac{\pi}{4}\right)}$$

$$= \frac{\pi}{4\times\frac{1}{\sqrt{2}}\sqrt{p}} = \frac{\pi}{2\sqrt{2}\sqrt{p}}$$

$$\therefore L\{F(t)\} = \frac{\pi}{2\sqrt{2}\sqrt{p}}$$

$$\Rightarrow F(t) = L^{-1}\left\{\frac{\pi}{2\sqrt{2}\sqrt{p}}\right\}$$

$$= \frac{\pi}{2\sqrt{2}}\,L^{-1}\left\{\frac{1}{\sqrt{p}}\right\}$$

$$= \frac{\pi}{2\sqrt{2}}\,\frac{1}{\sqrt{\pi t}} = \frac{\sqrt{\pi}}{2\sqrt{2t}} = \frac{1}{2}\sqrt{\frac{\pi}{2t}}$$

$$\therefore \int_0^\infty \cos t x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2t}} \quad \text{--- (1)}$$

(ii) Taking $t = 1$ in eqn (1)

$$\int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

(iii) $\int_0^\infty e^{-tx^2}\,dx$.

Let $F(t) = \int_0^\infty e^{-tx^2}\,dx$.

$$\therefore L\{F(t)\} = \int_0^\infty e^{-pt}F(t)\,dt$$

$$= \int_0^\infty e^{-pt}\left\{\int_0^\infty e^{-tx^2}\,dx\right\}dt$$

$$= \int_0^\infty \left[\int_0^\infty e^{-pt}\,e^{-tx^2}\,dt\right]dx, \text{ changing the order of integration}$$

$$= \Gamma(m)\Gamma(n)\frac{t^{m+n-1}}{\Gamma(m+n)}$$

putting $t=1$ in the above result, we get

$$B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}\,du = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \text{where } m>0,\ n>0. \quad (2)$$

Deduction:

Taking $u = \sin^2\theta$. $\quad u=0 \Rightarrow \theta=0$.

$du = 2\sin\theta\cos\theta\,d\theta$. $\quad u=1 \Rightarrow \theta=\pi/2$.

$\therefore$ from (2), we have

$$\int_0^{\pi/2} \sin^{2m-2}\theta\,(1-\sin^2\theta)^{n-1}\,2\sin\theta\cos\theta\,d\theta = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\Rightarrow 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n)$$

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta = \frac{1}{2}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{1}{2}B(m,n)$$

$\rightarrow$ find $L^{-1}\left\{\frac{1}{\sqrt{p}(p-a)}\right\}$ by the convolution theorem and deduce the value of $L^{-1}\left\{\frac{1}{p\sqrt{p+a}}\right\}$.

Sol$^n$:

Let $f(p) = \frac{1}{\sqrt{p}}$ and $g(p) = \frac{1}{p}$

Then, $F(t) = L^{-1}\left\{\frac{1}{\sqrt{p}}\right\} = L^{-1}\left\{\frac{1}{p^{1/2}}\right\} = \frac{t^{-1/2}}{\Gamma 1/2} = \frac{1}{\sqrt{\pi t}}$

and $G(t) = L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at}$.

Now, using the convolution theorem, we have

$$L^{-1}\{f(p)\cdot g(p)\} = \int_0^t F(u)\,G(t-u)\,du$$

$$L^{-1}\left\{\frac{1}{\sqrt{p}(p-a)}\right\} = \int_0^t \frac{1}{\sqrt{\pi}\sqrt{u}}e^{a(t-u)}\,du$$

$$= \frac{e^{at}}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{u}}e^{-au}\,du$$

put $au = x^2 \Rightarrow x=\sqrt{au}$ $\quad$ if $u=0$ then $x=0$; $u=t$ then $x=\sqrt{at}$

$a\,du = 2x\,dx \Rightarrow du = \frac{2x\,dx}{a}$ $\quad \Rightarrow \frac{du}{\sqrt{au}} = \frac{2dx}{a}$ $\quad \Rightarrow \frac{du}{\sqrt{u}} = \frac{2}{\sqrt{a}}dx$

$$= \frac{e^{at}}{\sqrt{\pi}}\int_0^{\sqrt{at}} e^{-x^2}\frac{2}{\sqrt{a}}\,dx$$

$$= \frac{e^{at}}{\sqrt{a}}\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{at}} e^{-x^2}\,dx = \frac{e^{at}}{\sqrt{a}}\,\mathrm{erf}(\sqrt{at})$$ (by defn of error function)

$$= \int_0^\infty L(e^{-tx^2})\,dx$$

$$= \int_0^\infty \frac{1}{p+x^2}\,dx \qquad \left[\because L\{e^{-at}\} = \frac{1}{p+a}\right]$$

$$= \int_0^\infty \frac{1}{(\sqrt{p})^2+x^2}\,dx$$

$$= \left[\frac{1}{\sqrt{p}}\tan^{-1}\left(\frac{x}{\sqrt{p}}\right)\right]_0^\infty$$

$$= \left[\frac{1}{\sqrt{p}}\frac{\pi}{2} - 0\right] = \frac{1}{\sqrt{p}}\frac{\pi}{2}.$$

$$\therefore L\{F(t)\} = \frac{1}{\sqrt{p}}\left(\frac{\pi}{2}\right)$$

$$F(t) = L^{-1}\left\{\frac{1}{\sqrt{p}}\frac{\pi}{2}\right\}$$

$$= \frac{\pi}{2} L^{-1}\left(\frac{1}{\sqrt{p}}\right)$$

$$F(t) = \frac{\pi}{2}\frac{1}{\sqrt{\pi t}} = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{t}}$$

$$\therefore \int_0^\infty e^{-tx^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{t}} \quad \text{—— (2)}$$

(vi) putting $t=1$ in eqn (2)

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

→ We know that $\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}\,dx$.

The complementary error function is defined as

$$\operatorname{erf} c(t) = 1 - \operatorname{erf}(t)$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\,dx - \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}\,dx$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\,dx + \frac{2}{\sqrt{\pi}}\int_t^0 e^{-x^2}\,dx \qquad \left[\because \operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\,dx. = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1 \quad \left(\because \int_0^\infty e^{-x^2}dx = 1\right) \Rightarrow \operatorname{erf}(\infty) = 1\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\int_t^0 e^{-x^2}\,dx + \int_0^\infty e^{-x^2}\,dx\right]$$

$$\boxed{\operatorname{erf} c(t) = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-x^2}\,dx.}$$

→ Show that $L^{-1}\left\{\frac{1}{p\sqrt{p+4}}\right\} = \frac{1}{2}\,\text{erf}(2\sqrt{t})$.

Sol$^n$: Let $f(p) = \frac{1}{p}$ and $g(p) = \frac{1}{\sqrt{p+4}}$

Then $F(t) = L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{p}\right\} = 1$

and $G(t) = L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{(p+4)^{1/2}}\right\} = e^{-4t} L^{-1}\left\{\frac{1}{p^{1/2}}\right\}$

$= e^{-4t}\frac{t^{-1/2}}{\Gamma\frac{1}{2}} = \frac{e^{-4t}}{\sqrt{\pi t}}$ $\left(\because \Gamma\frac{1}{2} = \sqrt{\pi}\right)$

i.e, $G(t) = \frac{e^{-4t}}{\sqrt{\pi t}}$

Now, using the convolution theorem.

$$L^{-1}\{f(p)\,g(p)\} = \int_0^t G(u)\,F(t-u)\,du$$

$$= \int_0^t e^{-4u}\frac{1}{\sqrt{\pi}\sqrt{u}}\cdot 1\,du$$

$$= \frac{1}{\sqrt{\pi}}\int_0^t e^{-4u}\frac{1}{\sqrt{u}}\,du.$$

put $4u = y^2 \Rightarrow y = 2\sqrt{u}$

$\Rightarrow du = \frac{1}{2}y\,dy$

$\Rightarrow du = \frac{1}{2}\cdot 2\sqrt{u}\,dy$

$\Rightarrow \frac{du}{\sqrt{u}} = dy$.

$$= \frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-y^2}\,dy$$

$$= \frac{1}{2}\,\frac{2}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-y^2}\,dy$$

$$= \frac{1}{2}\,\text{erf}(2\sqrt{t}).$$ $\left(\because \text{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx\right)$

→ Apply convolution theorem to show that

$$\int_0^t \sin u\cos(t-u)\,du = \frac{t}{2}\sin t.$$

Sol$^n$: ~~By~~ convolution theorem, we have

$$L\left\{\int_0^t F(u)\,G(t-u)\,du\right\} = L\{F(t)\}\,L\{G(t)\} \quad \text{(1)}$$

Take $F(t) = \sin t$ and $G(t) = \cos t$.

Then eq$^n$ (1) reduces to

$$L\left\{\int_0^t \sin u\cos(t-u)\,du\right\} = L\{\sin t\}\,L\{\cos t\}$$

$$= \frac{1}{p^2+1}\,\frac{p}{p^2+1} = \frac{p}{(p^2+1)^2}$$

$$\therefore \int_0^t \sin u\cos(t-u)\,du = L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\}.$$

$= \frac{1}{2} t \sin t$ $\left[ \text{w.k.T } L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{1}{2} t\sin t \right]$

$\therefore \int_0^t \sin u \cos(t-u)\, du = \frac{t}{2} \sin t$.

→ Evaluate $\int_0^t J_0(u)\, J_0(t-u)\, du$.

Sol^n: By convolution theorem,

$$L\left\{\int_0^t F(u)\, G(t-u)\, du\right\} = L\{F(t)\}\, L\{G(t)\} \quad \text{--- (1)}$$

Take $F(t) = J_0(t)$ and $G(t) = J_0(t)$.

Then eqn (1) reduces to

$$L\left\{\int_0^t J_0(u)\, J_0(t-u)\, du\right\} = L\{J_0(t)\}\, L\{J_0(t)\}$$

$$= \left(\frac{1}{\sqrt{p^2+1}}\right)^2 \quad \left(\because L\{J_0(t)\} = \frac{1}{\sqrt{p^2+1}}\right)$$

$$= \frac{1}{p^2+1}$$

$$\therefore \int_0^t J_0(u)\, J_0(t-u)\, du = L^{-1}\left(\frac{1}{p^2+1}\right)$$

$$= \sin t.$$

→ Apply convolution theorem to prove that

$$B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}\, dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}, \; m>0, n>0 \quad \text{(Beta function)}$$

Hence deduce that

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)}$$

Sol^n: By the convolution theorem,

$$L\left\{\int_0^t F(u)\, G(t-u)\, du\right\} = L\{F(t)\}\cdot G\{F(t)\} \quad \text{--- (1)}$$

Take $F(t) = t^{m-1}$ and $G(t) = t^{n-1}$.

Then eqn (1) reduces to

$$L\left\{\int_0^t u^{m-1}(t-u)^{n-1}\, du\right\} = L\{t^{m-1}\}\, L\{t^{n-1}\}.$$

$$= \frac{\Gamma(m)}{p^m}\,\frac{\Gamma(n)}{p^n} = \frac{\Gamma(m)\,\Gamma(n)}{p^{m+n}}$$

$$\therefore \int_0^t u^{m-1}(t-u)^{n-1}\, du = L^{-1}\left\{\frac{\Gamma(m)\,\Gamma(n)}{p^{m+n}}\right\}$$

$$= \Gamma(m)\,\Gamma(n)\; L^{-1}\left\{\frac{1}{p^{m+n}}\right\}$$

$$= \frac{1}{a}\int_0^t \cos^2 au \,\sin at\, du - \frac{1}{a}\int_0^t \sin au \cos au \cdot \cos at\, du.$$

$$= \frac{1}{2a}\sin at \int_0^t (1+\cos 2au)\, du - \frac{1}{2a}\cos at \int_0^t \sin 2au\, du$$

$$= \frac{1}{2a}\sin at \left[u + \frac{\sin 2au}{2a}\right]_0^t - \frac{1}{2a}\cos at\left[-\frac{\cos 2au}{2a}\right]_0^t$$

$$= \frac{1}{2a}\sin at\left[t + \frac{1}{2a}\sin 2at - 0\right] + \frac{1}{2a}\cos at \cdot \frac{1}{2a}\left(\cos 2at - 1\right)$$

$$= \frac{t}{2a}\sin at + \frac{1}{4a^2}\sin at \underbrace{\sin 2at}_{2\sin at\cos at} + \frac{1}{2a^2}\cos at\left(-\sin^2 at\right)$$

$$= \frac{t}{2a}\sin at + \frac{1}{2a^2}\cancel{\sin^2 at}\cos at - \frac{1}{2a^2}\cancel{\cos at \sin^2 at} \quad \left(\because \frac{1-\cos 2at}{2} = \sin^2 at\right)$$

$$= \frac{t\sin at}{2a} \quad \text{Ans.}$$

Q(vi) $L^{-1}\left\{\frac{1}{p(p^2+4)^2}\right\}$.

Sol: $L^{-1}\left\{\frac{1}{p(p^2+4)^2}\right\} = L^{-1}\left\{\frac{1}{p^2}\cdot\frac{p}{(p^2+4)^2}\right\}$.

Let $f(p) = \frac{1}{p^2}$ and $g(p) = \frac{p}{(p^2+4)^2}$

Then $F(t) = L^{-1}(f(p)) = L^{-1}\left(\frac{1}{p^2}\right) = t$

and $G(t) = L^{-1}\{g(p)\} = L^{-1}\left\{\frac{p}{(p^2+4)^2}\right\}$

$$= L^{-1}\left\{-\frac{1}{2}\frac{d}{dp}\left(\frac{1}{(p^2+4)}\right)\right\}$$

$$= -\frac{1}{2}L^{-1}\left\{\frac{d}{dp}\left(\frac{1}{(p^2+4)}\right)\right\}$$

$$= -\frac{1}{2}(-1)^1 t\, L^{-1}\left\{\frac{1}{(p^2+4)}\right\}$$

$$= \frac{1}{2}t\,\frac{1}{2}\sin 2t$$

$$= \frac{t}{4}\sin 2t.$$

Now, using the convolution theorem, we have

$$L^{-1}\{f(p)\,g(p)\} = \int_0^t G(u)\,F(t-u)\,du$$

$$= \int_0^t \frac{u}{4}\sin 2u\,(t-u)\,du.$$

$$= \frac{1}{4}\int_0^t (tu - u^2)\sin 2u\,du.$$

$$= \frac{1}{2} t \sin t \qquad \left[\text{w.k.T } L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{1}{2} t \sin t\right]$$

$$\therefore \int_0^t \sin u \cos(t-u)\, du = \frac{t}{2} \sin t.$$

→ Evaluate $\int_0^t J_0(u)\, J_0(t-u)\, du$.

Sol^n: By convolution theorem,

$$L\left\{\int_0^t F(u)\, G(t-u)\, du\right\} = L\{F(t)\}\, L\{G(t)\}. \quad \text{--- (1)}$$

Take $F(t) = J_0(t)$ and $G(t) = J_0(t)$.

Then eq^n (1) reduces to

$$L\left\{\int_0^t J_0(u)\, J_0(t-u)\, du\right\} = L\{J_0(t)\}\, L\{J_0(t)\}$$

$$= \left(\frac{1}{\sqrt{p^2+1}}\right)^2 \qquad \left(\because L\{J_0(t)\} = \frac{1}{\sqrt{p^2+1}}\right)$$

$$= \frac{1}{p^2+1}$$

$$\therefore \int_0^t J_0(u)\, J_0(t-u)\, du = L^{-1}\left(\frac{1}{p^2+1}\right)$$

$$= \sin t.$$

→ Apply convolution theorem to prove that

$$B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}\, dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}, \; m>0, n>0 \quad \text{(Beta function)}$$

Hence deduce that

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{1}{2} B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)}$$

Sol^n: By the convolution theorem,

$$L\left\{\int_0^t F(u)\, G(t-u)\, du\right\} = L\{F(t)\}\,.\, G\{F(t)\}. \quad \text{--- (1)}$$

Take $F(t) = t^{m-1}$ and $G(t) = t^{n-1}$.

Then eq^n (1) reduces to

$$L\left\{\int_0^t u^{m-1}(t-u)^{n-1}\, du\right\} = L\{t^{m-1}\}\, L\{t^{n-1}\}.$$

$$= \frac{\Gamma(m)}{p^m}\,\frac{\Gamma(n)}{p^n} = \frac{\Gamma(m)\,\Gamma(n)}{p^{m+n}}$$

$$\therefore \int_0^t u^{m-1}(t-u)^{n-1}\, du = L^{-1}\left\{\frac{\Gamma(m)\,\Gamma(n)}{p^{m+n}}\right\}$$

$$= \Gamma(m)\,\Gamma(n)\; L^{-1}\left\{\frac{1}{p^{m+n}}\right\}$$

On $x=t$ and the other end on $t=\infty$.
For this strip $x$ varies from 0 to $\infty$. Hence changing the order of integration — ① reduces to.

$$L\{H(t)\} = \int_{x=0}^{\infty} F(x)\left\{\int_{t=x}^{\infty} e^{-pt} G(t-x)\,dt\right\} dx$$

Putting $t-x=y$
$t = y+x$
$dt = dy$
limits for $y$: 0 to $\infty$.

$$= \int_{x=0}^{\infty} F(x)\left\{\int_{y=0}^{\infty} e^{-p(x+y)} G(y)\,dy\right\} dx$$

$$= \int_{x=0}^{\infty} F(x)\left\{e^{-px}\int_{y=0}^{\infty} e^{-py} G(y)\,dy\right\} dx$$

$$= \int_{x=0}^{\infty} e^{-px} F(x)\,dx \left[\int_{y=0}^{\infty} e^{-py} G(y)\,dy\right]$$

$$= \left[\int_{t=0}^{\infty} e^{-pt} F(t)\,dt\right]\left[\int_{t=0}^{\infty} e^{-pt} G(t)\,dt\right]$$ (By the Property of definite integral)

$$= L\{F(t)\}\cdot L\{G(t)\}$$

$$= f(p)\, g(p)$$

$$\therefore H(t) = L^{-1}\{f(p)\cdot G(p)\}$$

i.e. $L^{-1}\{f(p)\cdot g(p)\} = H(t)$

$$L^{-1}\{f(p)\, g(p)\} = \int_0^t F(x)\, G(x-t)\,dx$$

$$= F * G$$

<u>Note</u>: (1). The convolution theorem can be re-written as

$$L\left\{\int_0^t F(x)G(x-t)\,dx\right\} = L\{F(t) * G(t)\}$$
$$= L\{F(t)\}\, L\{G(t)\}$$

(2). while using the convolution theorem, we use one of the following two forms.

$$L^{-1}\{f(P)\cdot g(P)\} = \int_0^t F(x)\, G(t-x)\, dx \quad (\text{or})$$

$$L^{-1}\{f(P)\cdot g(P)\} = \int_0^t G(x) F(t-x) dx.$$

→ Use the convolution theorem to find

(i) $L^{-1}\left\{\frac{1}{(P+a)(P+b)}\right\}$ (ii) $L^{-1}\left\{\frac{1}{(P+1)(P-2)}\right\}$ (iii) $L^{-1}\left\{\frac{1}{(P+1)(P-1)}\right\}$ (iv) $L^{-1}\left\{\frac{1}{(P-1)(P+2)}\right\}$

(v) $L^{-1}\left\{\frac{P}{(P^2+a^2)^2}\right\}$ (vi) $L^{-1}\left\{\frac{1}{P(P^2+4)^2}\right\}$ (vii) $L^{-1}\left\{\frac{1}{(P-2)(P^2+1)}\right\}$

Sol'n: (i) $L^{-1}\left\{\frac{1}{(P+a)(P+b)}\right\}$

Let $f(P) = \frac{1}{P+a}$ ; $g(P) = \frac{1}{P+b}$

then, $F(t) = L^{-1}\{f(P)\} = L^{-1}\left\{\frac{1}{P+a}\right\} = e^{-at}$

and $G(t) = L^{-1}\{g(P)\} = L^{-1}\left\{\frac{1}{P+b}\right\} = e^{-bt}$

Now using the convolution theorem, we have

$$L^{-1}\{f(P)g(P)\} = \int_0^t F(u)G(t-u)du$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(P+a)}\,\frac{1}{(P+b)}\right\} = \int_0^t e^{-au}\, e^{-b(t-u)}\, du$$

$$= e^{-bt}\int_0^t e^{(b-a)u}\, du$$

$$= e^{-bt}\left[\frac{e^{(b-a)u}}{b-a}\right]_0^t = \frac{e^{-bt}}{b-a}\left[e^{(b-a)t}-1\right] = \frac{1}{(b-a)}\left[e^{-at}-e^{-bt}\right]$$

$$\therefore L^{-1}\left\{\frac{1}{P+a}\,\frac{1}{P+b}\right\} = \frac{1}{(b-a)}\left\{e^{-at}-e^{-bt}\right\}$$

→ $L^{-1}\left\{\frac{P}{(P^2+a^2)^2}\right\}$.

Sol'n: Let $f(P) = \frac{P}{P^2+a^2}$ and $g(P) = \frac{1}{P^2+a^2}$

Then $F(t) = L^{-1}\{f(P)\} = L^{-1}\left\{\frac{P}{P^2+a^2}\right\} = \cos at$

and $G(t) = L^{-1}\{g(P)\} = L^{-1}\left\{\frac{1}{P^2+a^2}\right\} = \frac{1}{a}\sin at$

Now, using the convolution theorem, we have

$$L^{-1}\{f(P)g(P)\} = \int_0^t F(u)G(t-u)du$$

$$L^{-1}\left\{\frac{P}{(P^2+a^2)^2}\right\} = \int_0^t \cos au\cdot\frac{1}{a}\sin a(t-u)\, du$$

$$= \frac{1}{a}\int_0^t \cos au\left[\sin at\cos au - \cos at\sin au\right] du.$$

# Convolution

→ Another important general Property of the Laplace transform has to do with products of transforms. It often happens that we are given two transforms $f(p)$ and $g(p)$ whose inverses $F(t)$ and $G(t)$. (i.e. $L^{-1}\{f(p)\} = F(t)$ & $L^{-1}\{g(p)\} = G(t)$).

we would like to calculate the inverse of the product $h(p) = f(p).g(p)$ (i.e. $L^{-1}\{h(p)\} = L^{-1}\{f(p)\,g(p)\}$). From those known inverses $F(t)$ and $G(t)$. This inverse $H(t)$ is denoted by $(F*G)(t)$ and is called Convolution of $F(t)$ and $G(t)$.

→ Let $F(t)$ and $G(t)$ be two functions of class A then the convolution of the two functions $F(t)$ and $G(t)$ denoted by $F*G$ and is defined as

$$F*G = \int_0^t F(x)\,G(t-x)\,dx.$$

→ Properties of Convolution:

(i) $F*G = G*F$
(i.e, Commutative)

Sol$^n$: $F*G = \int_0^t F(x)\,G(t-x)\,dx$ Putting $t-x=y$
$\Rightarrow x = t-y$
$dx = -dy$

$$= \int_t^0 F(t-y)\,G(y)\,(-dy)$$

$$= -\int_t^0 F(t-y)\,G(y)\,dy$$

$$= \int_0^t G(y)\,F(t-y)\,dy$$

$$= G*F$$

→ (ii) $(F*G)*H = F*(G*H)$

(iii) $F*(G+H) = (F*G)+(F*H)$

Sol'n: $F*(G+H) = \int_0^t F(x)\,(G+H)(t-x)\,dx$

$$= \int_0^t F(x)\left[G(t-x)+H(t-x)\right]dx$$

$$= \int_0^t F(x)\,G(t-x)\,dx + \int_0^t F(x)\,H(t-x)\,dx$$

$$= (F*G) + (F*H)$$

→ Convolution theorem: (Convolution Property):—

Let $F(t)$ and $G(t)$ be two functions of class A and let $L^{-1}\{f(P)\} = F(t)$ and $L^{-1}\{g(P)\} = G(t)$ then

$$L^{-1}\{f(P)\cdot g(P)\} = \int_0^t F(x)\,G(t-x)\,dx = F*G$$

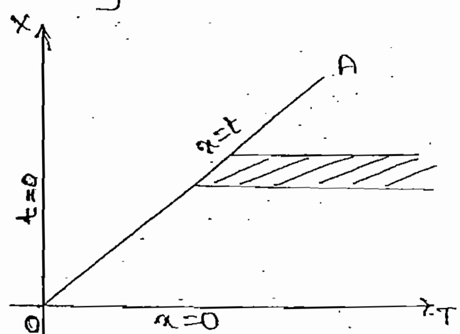
Proof: We have to show that

$$L\left\{\int_0^t F(x)\,G(t-x)\,dx\right\} = f(P)\cdot g(P)$$

Let $H(t) = \int_{x=0}^{t} F(x)\,G(t-x)\,dx = F*G.$

Since $L\{H(t)\} = \int_{t=0}^{\infty} e^{-Pt}\,H(t)\,dt$ (by definition of L.T)

$$\therefore L\{H(t)\} = \int_{t=0}^{\infty} e^{-Pt}\left[\int_{x=0}^{t} F(x)\,G(t-x)\,dx\right]dt \quad \text{——— (1)}$$




In equation (1), the region of integration in the double integral is the infinite area below the line OA (with equation $x=t$) and above the line OT (with equation $x=0$). Here $t$ & $x$ are measured along OT and OX respectively.

To change the order of Integration:

Draw an elementary strip parallel to T-axis. One end lies

$= \cos t + 1 = 1 - \cos t = F_1(t)$, say

Let $L^{-1}\left\{\frac{1}{p^2(p^2+1)}\right\}$.

Clearly it is in the form of $L^{-1}\left\{\frac{f_1(p)}{p}\right\}$

[where $f_1(p) = \frac{1}{p(p^2+1)}$

and $L^{-1} f_1(p) = L^{-1}\left\{\frac{1}{p(p^2+1)}\right\} = 1 - \cos t = F_1(t)$]

$\therefore L^{-1}\left\{\frac{1}{p^2(p^2+1)}\right\} = \int_0^t F_1(x)\,dx$

$= \int_0^t (1-\cos x)\,dx$

$= [x - \sin x]_0^t$

$= t - \sin t$

Now let $L^{-1}\left\{\frac{1}{p^3(p^2+1)}\right\}$

clearly it is in the form of $L^{-1}\left\{\frac{f_2(p)}{p}\right\}$

where $f_2(p) = \frac{1}{p^2(p^2+1)}$

and $L^{-1}(f_2(p)) = t - \sin t = F_2(t)$, say

$\therefore L^{-1}\left\{\frac{1}{p^3(p^2+1)}\right\} = \int_0^t F_2(x)\,dx$

$= \int_0^t (x - \sin x)\,dx$

$= \left[\frac{x^2}{2} + \cos x\right]_0^t$

$= \frac{t^2}{2} + \cos t - 1$

→ Find $L^{-1}\left\{\frac{1}{p(p^2+1)}\right\}$

→ Find $L^{-1}\left\{\frac{1}{p}\log\frac{p+2}{p+1}\right\}$

Sol'n: $f(p) = \log\frac{p+2}{p+1}$

$= \log(p+2) - \log(p+1)$

$f'(p) = \frac{1}{p+2} - \frac{1}{p+1}$

$\therefore L^{-1}\{f'(p)\} = L^{-1}\left\{\frac{1}{p+2} - \frac{1}{p+1}\right\}$

$-t L^{-1}\{f(p)\} = L^{-1}\left(\frac{1}{p+2}\right) - L^{-1}\left(\frac{1}{p+1}\right)$

$= e^{-2t} - e^{-t}$ $\left[\because L^{-1} f'(p) = (-1)^1 t\, L^{-1}\{f(p)\}\right]$

$\Rightarrow L^{-1}\{f(p)\} = \frac{e^{-t} - e^{-2t}}{t} = F(t)$, say

$\therefore L^{-1}\left\{\frac{1}{p} f(p)\right\} = \int_0^t F(x)\,dx$

$= \int_0^t \frac{e^{-x} - e^{-2x}}{x}\,dx$

$\therefore L^{-1}\left\{\frac{1}{p} f(p)\right\} = \int_0^t \left[\frac{e^{-x} - e^{-2x}}{x}\right] dx$

→ Find $L^{-1}\left\{\frac{1}{p}\left\{\log\left(1+\frac{1}{p^2}\right)\right\}\right\}$

→ Find $L^{-1}\left[\frac{1}{p(p+1)^3}\right]$

Sol'n: Given $L^{-1}\left\{\frac{1}{p(p+1)^3}\right\}$

Clearly it is in the form of $L^{-1}\left\{\frac{f(p)}{p}\right\}$

Let $f(p) = \frac{1}{(p+1)^3}$

$$\Rightarrow L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{(p+1)^3}\right\}$$

$$= e^{-t} L^{-1}\left\{\frac{1}{p^3}\right\}$$

$$= e^{-t}\frac{t^2}{2} = F(t), \text{ say}$$

Now $L^{-1}\left\{\frac{1}{p(p+1)^3}\right\} = \int_0^t F(x)\,dx$

$$= \int_0^t e^{-x}\frac{x^2}{2}\,dx$$

$$= \frac{1}{2}\int_0^t e^{-x} x^2\,dx$$

$$= \frac{1}{2}\left[\left(-e^{-x}x^2\right)_0^t + 2\int_0^t e^{-x} x\,dx\right]$$

$$= \frac{1}{2}\left\{\left[-e^{-t}t^2+0\right] + 2\left[-e^{-x}x\right]_0^t - 2\left[e^{-x}\right]_0^t\right\}$$

$$= \frac{1}{2}e^{-t}t^2 + (-e^{-t}t+0) - (e^{-t}-1)$$

$$= -\frac{1}{2}e^{-t}t^2 - e^{-t}t - e^{-t}+1$$

$$= 1-e^{-t}\left[\frac{t^2}{2}+t+1\right]$$

→ If $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{t}{2}\sin t$,

find $L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\}$.

Sol'n: $L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\} = L^{-1}\left\{\frac{1}{p}\cdot\frac{p}{(p^2+1)^2}\right\}$

which is in the form of $L^{-1}\left\{\frac{f(p)}{p}\right\}$

Here $f(p) = \frac{p}{(p^2+1)^2}$

and we know that

$$L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{t}{2}\sin t = F(t), \text{ say}$$

$$L^{-1}\left\{\frac{1}{p}\cdot\frac{p}{(p^2+1)^2}\right\} = \int_0^t F(x)\,dx$$

$$= \int_0^t \frac{x}{2}\sin x\,dx$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\} = \frac{1}{2}\left[-x\cos x+\sin x\right]_0^t$$

$$= \frac{1}{2}(\sin t - t\cos t)$$

→ Find $L^{-1}\left\{\frac{e^{-\sqrt{p}}}{p}\right\}$, and hence deduce that

$L^{-1}\left\{\frac{e^{-x\sqrt{p}}}{p}\right\} = \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)$.

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Soln: Let $f(p) = e^{-\sqrt{p}}$

$\Rightarrow L\{F(t)\} = e^{-\sqrt{p}}$

$\Rightarrow F(t) = L^{-1}\{e^{-\sqrt{p}}\}$

$= L^{-1}\left\{1-\sqrt{p}+\frac{(\sqrt{p})^2}{2!}-\frac{(\sqrt{p})^3}{3!}+\frac{(\sqrt{p})^4}{4!}-\cdots\right\}$

$= L^{-1}\left[1-\sqrt{p}+\frac{p}{2!}-\frac{p^{3/2}}{3!}+\frac{p^2}{4!}-\cdots\right]$

$= L^{-1}\{1\}-L^{-1}\{p^{1/2}\}+\frac{1}{2!}L^{-1}\{p\}-\frac{1}{3!}L^{-1}\{p^{3/2}\}$

$+\frac{1}{4!}L^{-1}\{p^2\}+\cdots$

$= L^{-1}\{1\}-L^{-1}\left\{\frac{1}{p^{-1/2}}\right\}+\frac{1}{2!}L^{-1}\left\{\frac{1}{p^{-1}}\right\}-\frac{1}{3!}L^{-1}\left\{\frac{1}{p^{-3/2}}\right\}$

$+\frac{1}{4!}L^{-1}\left\{\frac{1}{p^{-2}}\right\}-\cdots$

$=$

→ **Heaviside's expansion theorem (or formula).** (31)

Let $F(p)$ and $G(p)$ be two polynomials in $p$ where $F(p)$ has degree less than that of $G(p)$. If $G(p)$ has 'n' distinct zeros $\alpha_r$, $(r = 1, 2, \ldots, n)$.

i.e, $G(p) = (p-\alpha_1)(p-\alpha_2)\ldots\ldots(p-\alpha_n)$. then

$$L^{-1}\left\{\frac{F(p)}{G(p)}\right\} = \sum_{r=1}^{n} \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t}.$$

Proof: Since $F(p)$ is a polynomial of degree less than that of $G(p)$ and $G(p)$ has $n$ distinct zeros $\alpha_r$, $r = 1, 2, \ldots n$

$$\therefore \frac{F(p)}{G(p)} = \frac{F(p)}{(p-\alpha_1)(p-\alpha_2)\ldots\ldots(p-\alpha_n)}$$

$$= \frac{A_1}{p-\alpha_1} + \frac{A_2}{p-\alpha_2} + \ldots\ldots + \frac{A_r}{p-\alpha_r} + \ldots\ldots + \frac{A_n}{p-\alpha_n}.$$

To compute $A_r$, multiplying both sides by $p-\alpha_r$.

i.e, $$\frac{F(p)}{G(p)}\cdot(p-\alpha_r) = (p-\alpha_r)\left[\frac{A_1}{p-\alpha_1} + \frac{A_2}{p-\alpha_2} + \ldots\ldots + \frac{A_r}{p-\alpha_r} + \ldots\ldots + \frac{A_n}{p-\alpha_n}\right]$$

Taking limit as $p \to \alpha_r$, we get

$$\Rightarrow \lim_{p\to\alpha_r} \frac{F(p)}{G(p)}(p-\alpha_r) = A_r.$$

$$\Rightarrow A_r = \lim_{p\to\alpha_r} \frac{F(p)}{G(p)}(p-\alpha_r)$$

$$= F(\alpha_r) \lim_{p\to\alpha_r} \frac{(p-\alpha_r)}{G(p)} \quad \left(\text{Form } \frac{0}{0}\right)$$

$$= F(\alpha_r) \lim_{p\to\alpha_r} \frac{1}{G'(p)}$$

where $G(p) = (p-\alpha_1)\cdot(p-\alpha_2)\ldots(p-\alpha_n)$

$$= F(\alpha_r) \frac{1}{G'(\alpha_r)}$$

$$\therefore \frac{F(p)}{G(p)} = \frac{F(\alpha_1)}{G'(\alpha_1)}\frac{1}{(p-\alpha_1)} + \frac{F(\alpha_2)}{G'(\alpha_2)}\frac{1}{(p-\alpha_2)} + \frac{F(\alpha_3)}{G'(\alpha_3)}\frac{1}{(p-\alpha_3)} + \ldots$$

$$\ldots + \frac{F(\alpha_r)}{G'(\alpha_r)}\frac{1}{(p-\alpha_r)} + \ldots\ldots + \frac{F(\alpha_n)}{G'(\alpha_n)}\frac{1}{(p-\alpha_n)}$$

Hence $$L^{-1}\left\{\frac{F(p)}{G(p)}\right\} = \frac{F(\alpha_1)}{G'(\alpha_1)} L^{-1}\left(\frac{1}{p-\alpha_1}\right) + \frac{F(\alpha_2)}{G'(\alpha_2)} L^{-1}\left(\frac{1}{p-\alpha_2}\right)$$

$$+ \ldots + \frac{F(\alpha_r)}{G'(\alpha_r)} L^{-1}\left(\frac{1}{p-\alpha_r}\right) + \ldots + \frac{F(\alpha_n)}{G'(\alpha_n)} L^{-1}\left(\frac{1}{p-\alpha_n}\right)$$

$$y(t) = L^{-1}\left\{\frac{p}{(p^2+4)(p^2+9)}\right\} + L^{-1}\left\{\frac{p}{p^2+9}\right\} + A\,L^{-1}\left\{\frac{1}{p^2+9}\right\}$$

$$= \frac{1}{5}L^{-1}\left\{\frac{p}{(p^2+4)} - \frac{p}{(p^2+9)}\right\} + L^{-1}\left\{\frac{p}{p^2+9}\right\} + \frac{A}{3}L^{-1}\left\{\frac{1}{p^2+9}\right\}$$

$$= \frac{1}{5}L^{-1}\left\{\frac{p}{p^2+4}\right\} - \frac{1}{5}L^{-1}\left\{\frac{p}{p^2+9}\right\} + \cos 3t + \frac{A}{3}\sin 3t$$

$$y(t) = \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{A}{3}\sin 3t$$

$$\Rightarrow y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t$$

$$y(t) = \frac{1}{5}\cos \pi + \frac{4}{5}\cos\left(3\frac{\pi}{2}\right) + \frac{A}{3}\sin\left(\frac{3\pi}{2}\right) \quad \left(\because y\left(\frac{\pi}{2}\right) = -1\right)$$

$$-1 = \frac{1}{5}(-1) - \frac{1}{5}(0) + 0 + \frac{A}{3}(-1)$$

$$\Rightarrow -1 = -\frac{1}{5} - \frac{A}{3} \Rightarrow -\frac{4}{5} = -\frac{A}{3}$$

$$\Rightarrow A = \frac{12}{5}$$

$\therefore$ The required solution is

$$y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{12}{5\times 3}\sin 3t$$

$$y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t.$$

→ $(D^2+3D+2)y = 0$ ; $y = y_0$ and $Dy = y_1$ at $t=0$

→ solve $(D^2-D-2)y = 20\sin 2t$ ; $y=-1$, $Dy = 2$ when $t=0$

→ Solve $(D+1)^2 y = t$ ; $y=-3$ when $t=0$ and $y=-1$ when $t=1$

→ Solve $(D^2+2D+1)y = 3te^{-t}$, $t>0$ subject to the conditions $y=4$, $Dy=2$ when $t=0$

→ Solve $(D^2+1)y = t\cos 2t$, $y=0$, $\frac{dy}{dt}=0$ when $t=0$

→ solve $(D^2-3D+2)y = 1-e^{2t}$, $y=1$, $Dy=0$ when $t=0$

→ Solve $(D^2+1)y = \sin t \sin 2t$, $t>0$ if $y=1$, $Dy=0$ when $t=0$

→ solve $(D^3-D)y = 2\cos t$, $y=3$, $Dy=2$, $D^2y=1$ when $t=0$

## Solution of ordinary Differential Equations with Constant Coefficients:

The Laplace transform is very useful in solving ordinary linear differential eqns. with constant coefficients.

Suppose we wish to solve the $n^{th}$ order ordinary linear differential eqn with constant coefficients

$$\frac{d^ny}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + a_2 \frac{d^{n-2}y}{dt^{n-2}} + \cdots + a_n y = F(t) \quad (1)$$

where $F(t)$ is a function of the independent variable $t$. and $a_0, a_1, a_2, \ldots a_n$ are constants, subject to the initial conditions

$$y(0) = k_0, \; y'(0) = k_1, \; y''(0) = k_2, \ldots \; y^{n-1}(0) = k_{n-1} \quad (2)$$

where $k_0, k_1, \ldots k_{n-1}$ are constants.

On taking the Laplace transform of bothsides of eqn (1) and using conditions (2), we obtain an algebraic equation known as "subsidiary eqn" for determination of $L\{y(t)\}$. The required solution is then obtained by finding the inverse Laplace transform of $L\{y(t)\} = f(p)$.

Notations

→ $\frac{dy}{dt} = Dy = y'(t) = y^{(1)}(t)$;

$\frac{d^2y}{dt^2} = D^2y = y''(t) = y^{(2)}(t)$; ..... $\frac{d^ny}{dt^n} = D^ny = y^n(t)$ etc.

→ At $t=0$, we have

$y(0) = y_0, y'(0) = y_1, y''(0) = y_2, \ldots y^{(n)}(0) = y_n$.

→ Solve $\frac{d^2y}{dt^2}+y=0$ under the conditions that $y=1$, $\frac{dy}{dt}=0$ when $t=0$.

Solⁿ: Given that $\frac{d^2y}{dt^2}+y=0$, i.e. $y''+y=0$ — (1)

Taking Laplace transform of both sides of eqⁿ(1),

we get

$$L(y'')+L(y)=L(0)$$

$$p^2L\{y(t)\}-py(0)-y'(0)+L\{y(t)\}=0$$

$$p^2L\{y(t)\}-p(1)-0+L\{y(t)\}=0 \quad [\because y(0)=1 \ \& \ y'(0)=0]$$

$$(p^2+1)L\{y(t)\}=p$$

$$\Rightarrow L\{y(t)\}=\frac{p}{p^2+1}$$

Taking inverse Laplace transform, we get

$$\Rightarrow y(t)=L^{-1}\left\{\frac{p}{p^2+1}\right\}$$

$$=\cos t.$$

$\therefore$ $y(t)=\cos t$. which is the required solution.

→ Solve $(D^2+m^2)x=a\cos nt$, $t>0$.
$x=x_0$ and $Dx=x_1$, when $t=0$, $n\neq m$

Solⁿ Given eqⁿ is

$$(D^2+m^2)x=a\cos nt.$$

$$x''+m^2x=a\cos nt \quad \text{— (1)}$$

Taking Laplace transform of both sides of (1)

we get

$$L\{x''\}+L\{m^2x\}=L\{a\cos nt\}$$

$$p^2L\{x(t)\}-p\{x(0)\}-x'(0)+m^2Lx(t)=\frac{ap}{p^2+n^2} \quad \text{— (2)}$$

Using the given conditions $x=x_0$ and $Dx=x_1$ when $t=0$

eqⁿ(2) reduces to

$$p^2L\{x(t)\}-px_0-x_1+m^2L\,x(t)=\frac{ap}{p^2+n^2}$$

$$\Rightarrow (p^2+m^2)L\{x(t)\}=\frac{ap}{p^2+n^2}+px_0+x_1$$

$$\Rightarrow L\{x(t)\} = \frac{p}{p^2+m^2}x_0 + \frac{1}{p^2+m^2}x_1 + \frac{a\cdot p}{(p^2+m^2)(p^2+n^2)}$$

$$= x_0\frac{p}{p^2+m^2} + x_1\cdot\frac{1}{p^2+m^2} + \frac{a}{m^2-n^2}\left[\frac{-p}{p^2+m^2}+\frac{p}{p^2+n^2}\right]$$

Taking the inverse Laplace transform, we get

$$x(t) = x_0 L^{-1}\left\{\frac{p}{p^2+m^2}\right\} + x_1 L^{-1}\left\{\frac{1}{p^2+m^2}\right\} + \frac{a}{m^2-n^2}\left[L^{-1}\left\{\frac{-p}{p^2+m^2}\right\} + L^{-1}\left\{\frac{p}{p^2+n^2}\right\}\right].$$

$$= x_0\cos mt + x_1\frac{1}{m}\sin mt + \frac{a}{m^2-n^2}(-\cos mt + \cos nt)$$

$$x(t) = x_0\cos mt + \frac{x_1}{m}\sin mt + \frac{a}{m^2-n^2}\left[-\cos mt + \cos nt\right]$$

→ Solve $(D+2)^2 y = 4e^{-2t}$, $y(0) = -1$ & $y'(0) = 4$.
$(p^2+4D+4)y = 4e^{-2t}$

→ Solve $(D^2-2D+2)y = 0$, $y = Dy = 1$, when $t=0$

→ Solve $(D^2+4D+4)x = \sin wt$, $t>0$ with $x_0$ and $x_1$ for values of $x$ and $Dx$ when $t=0$ (i.e., $x = x_0$ & $Dx = x_1$, when $t=0$)

→ Solve $(D^2+9)y = \cos 2t$ if $y(0) = 1$, $y(\pi/2) = -1$.

Sol$^n$: Given eqn $(D^2+9)y = \cos 2t$ —— (1)

Taking Laplace transform of both sides of eqn (1), we get

$$L\{y''\} + 9L\{y\} = L\{\cos 2t\}$$

$$\Rightarrow p^2 L\{y(t)\} - p\,y(0) - y'(0) + 9L\{y(t)\} = \frac{p}{p^2+4} \quad \text{—— (2)}$$

Using the given condition $y(0) = 1$ eqn (2) reduces to

$$p^2 L\{y(t)\} - p(1) - A + 9L\{y(t)\} = \frac{p}{p^2+4}$$

(Taking $y'(0) = A$ constant)

$$\Rightarrow (p^2+9)L\{y(t)\} = \frac{p}{p^2+4} + p + A.$$

$$\Rightarrow L\{y(t)\} = \frac{p}{(p^2+4)(p^2+9)} + \frac{p}{p^2+9} + A\cdot\frac{1}{p^2+9}.$$

Taking inverse Laplace transform, we get

$$y(t) = L^{-1}\left\{\frac{p}{(p^2+4)(p^2+9)}\right\} + L^{-1}\left\{\frac{p}{p^2+9}\right\} + A\, L^{-1}\left\{\frac{1}{p^2+9}\right\}$$

$$= \frac{1}{5} L^{-1}\left\{\frac{p}{(p^2+4)} - \frac{p}{(p^2+9)}\right\} + L^{-1}\left\{\frac{p}{p^2+9}\right\} + \frac{A}{3} L^{-1}\left\{\frac{1}{p^2+9}\right\}$$

$$= \frac{1}{5} L^{-1}\left\{\frac{p}{p^2+4}\right\} - \frac{1}{5} L^{-1}\left\{\frac{p}{p^2+9}\right\} + \cos 3t + \frac{A}{3}\sin 3t$$

$$y(t) = \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{A}{3}\sin 3t$$

$$\Rightarrow y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t$$

$$y(t) = \frac{1}{5}\cos \pi + \frac{4}{5}\cos\left(3\frac{\pi}{2}\right) + \frac{A}{3}\sin\left(\frac{3\pi}{2}\right) \quad \left(\because y\left(\frac{\pi}{2}\right) = -1\right)$$

$$-1 = \frac{1}{5}(-1) - \frac{1}{5}(0) + 0 + \frac{A}{3}(-1)$$

$$\Rightarrow -1 = -\frac{1}{5} - \frac{A}{3} \Rightarrow -\frac{4}{5} = -\frac{A}{3}$$

$$\Rightarrow A = \frac{12}{5}$$

$\therefore$ The required solution is

$$y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{12}{5\times 3}\sin 3t$$

$$\underline{\underline{y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t}}$$

→ $(D^2+3D+2)y = 0$ ; $y = y_0$ and $Dy = y_1$ at $t=0$

→ solve $(D^2-D-2)y = 20\sin 2t$ ; $y=-1$, $Dy=2$ when $t=0$

→ Solve $(D+1)^2 y = t$ ; $y=-3$ when $t=0$
and $y=-1$ when $t=1$

→ solve $(D^2+2D+1)y = 3te^{-t}$, $t>0$
subject to the conditions $y=4$, $Dy=2$ when $t=0$

→ Solve $(D^2+1)y = t\cos 2t$, $y=0$, $\frac{dy}{dt}=0$ when $t=0$

→ solve $(D^2-3D+2)y = 1-e^{2t}$, $y=1$, $Dy=0$ when $t=0$

→ solve $(D^2+1)y = \sin t \sin 2t$, $t>0$
if $y=1$, $Dy=0$ when $t=0$

→ solve $(D^3-D)y = 2\cos t$, $y=3$, $Dy=2$, $D^2y=1$ when $t=0$

(34)

→ Solve $(D^3 - D^2 - D + 1)y = 8te^{-t}$
if $y = D^2y = 0$, $Dy = 1$ when $t = 0$

→ Solve $(D^4 - 1)y = 1$, $y = Dy = D^2y = D^3y = 0$ at $t = 0$

→ Solve $(D^2 + D)y = t^2 + 2t$. where $y(0) = 4$, $y'(0) = -2$

→ Solve $(D^4 + 2D^2 + 1)y = 0$ where $y(0) = 0, y'(0) = 1, y''(0) = 2$
and $y'''(0) = -3$.

→ Solve $(D^3 + 1)y = 1$, $t > 0$
$y = Dy = D^2y = 0$ when $t = 0$

→ Solve $(D^2 + n^2)y = a\sin(nt + \alpha)$, $y = 0$, $Dy = 0$ when $t = 0$

**To form the required differential equation.** The general solution of the required differential equation may be written as

$$y = Ay_1 + By_2 = Ax^2 + Bx^2 \log x. \qquad \ldots(1)$$

where $A$ and $B$ are arbitrary constants.

Differentiating (1), $\quad y' = 2Ax + B\,(2x \log x + x). \qquad \ldots(2)$

Differentiating (2), $\quad y'' = 2A + B\,(2 \log x + 2 + 1). \qquad \ldots(3)$

We now eliminate $A$ and $B$ from (1), (2) and (3). To this end, we first solve (2) and (3) for $A$ and $B$. Multiplying both sides of (3) by $x$, we get

$$xy'' = 2Ax + B\,(3x + 2x \log x). \qquad \ldots(4)$$

Subtracting (2) from (4), $\quad xy'' - y' = 2Bx \quad$ or $\quad B = (xy'' - y')/2x$.

Substituting this value of $B$ in (3), we have

$$2A = y'' - (1/2x)\,(xy'' - y')\,(3 + 2 \log x)$$

or
$$A = (1/4x)\,[2xy'' - (xy'' - y')\,(3 + 2 \log x)].$$

Substituting the above values $A$ and $B$ in (1), we have

$$y = (x/4)\,[2xy'' - 3xy'' + 3y' - 2xy'' \log x + 2y' \log x] + (x/2) \log x\,(xy'' - y')$$

or $\quad 4y = x\,(-xy'' + 3y' - 2xy'' \log x + 2y' \log x) + 2x \log x\,(xy'' - y')$

or $\quad x^2y'' - 3xy' + 4y = 0$, which is the required equation.

**Ex. 8.** *Evaluate the Wronskian of the functions $x$ and $x\,e^x$. Hence conclude whether or not these are linearly independent. If they are independent, set up the differential equation having them as its independent solutions.*

[Meerut 97]

**Sol.** Let $y_1 = x$ and $y_2 = x\,e^x$. Then their Wronskian $W(x)$ is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x\,e^x \\ 1 & e^x + x\,e^x \end{vmatrix} = x\,e^x + x^2\,e^x - x\,e^x = x^2\,e^x,$$

which is not identically equal to zero on $(-\infty, \infty)$. Hence $y_1$ and $y_2$ are linearly independent.

**To form the required differential equation.** The general solution of the required differential equation may be written as

$$y = Ay_1 + By_2 = Ax + Bx\,e^x, \qquad \ldots(1)$$

where $A$ and $B$ are arbitrary constants.

Differentiating (1), $\quad y' = A + B\,(e^x + xe^x) = A + B\,(1 + x)e^x. \qquad \ldots(2)$

Differentiating (2), $\quad y'' = B[e^x + (1 + x)e^x] = Be^x(2 + x). \qquad \ldots(3)$

We now eliminate $A$ and $B$ from (1), (2) and (3). To this we first solve (2) and (3) for $A$ and $B$.

From (3), $\quad B = y''/[e^x(2 + x)].$

Substituting this value of $B$ in (2), we have

$$A = y' - B\,(1 + x)e^x = y' - \frac{1+x}{2+x}y'' = \frac{(2+x)y' - (1+x)y''}{2+x}.$$

# * Independence of Solutions of Linear Differential Equations * Wronskian and Its Properties

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## SOLVED EXAMPLES

**Ex. 1.** *If $y_1(x) = \sin 3x$ and $y_2(x) = \cos 3x$ are two solutions of differential equation $y'' + 9y = 0$, show that $y_1(x)$ and $y_2(x)$ are linearly independent solutions.* **[Delhi B.Sc. (Hons) 1996]**

**Sol.** The Wronskian of $y_1(x)$ and $y_2(x)$

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin 3x & \cos 3x \\ 3\cos 3x & -3\sin 3x \end{vmatrix} = -3\sin^2 3x - 3\cos^2 3x$$

$$= -3(\sin^2 3x + \cos^2 3x) = -3 \neq 0.$$

Since $W(x) \neq 0$, $y_1(x)$ and $y_2(x)$ are linearly independent solutions of $y'' + 9y = 0$.

**Ex. 2.** *Prove that $\sin 2x$ and $\cos 2x$ are solutions of the differential equation $y'' + 4y = 0$ and these solutions are linearly independent.* **[Delhi (B.Sc.) (G) 1998]**

**Sol.** Given equation is $\quad y'' + 4y = 0.$ ...(1)

Let $\quad y_1(x) = \sin 2x$ and $y_2(x) = \cos 2x$ ...(2)

Now, $\quad y_1' = 2\cos 2x$ and $y_1'' = -4\sin 2x$ ...(3)

$$y_1''(x) + 4y_1(x) = -4\sin 2x + 4\sin 2x = 0, \text{ by (2) and (3)}$$

Hence, $y_1(x) = \sin 2x$ is a solution of (1). Similarly we can prove that $y_2(x)$ is a solution of (1).

Now, the Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} = -2\sin^2 2x - 2\cos^2 2x$$

$$= -2(\sin^2 2x + \cos^2 2x) = -2 \neq 0.$$

Since $W(x) \neq 0$, $\sin 2x$ and $\cos 2x$ are linearly independent solutions of (1).

**Ex. 3.** *Show that linearly independent solutions of $y'' - 2y' + 2y = 0$ are $e^x \sin x$ and $e^x \cos x$. What is the general solution? Find the solution $y(x)$ with the property $y(0) = 2$, $y'(0) = 3$.* **[Delhi B.Sc. (P) 96, Delhi B.Sc. (H) 2002]**

**Sol.** Given equation is $\quad y'' - 2y' + 2y = 0.$ ...(1)

Let $\quad y_1(x) = e^x \sin x$ and $y_2(x) = e^x \cos x$. ...(2)

From (2), $\quad y_1'(x) = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$ ...(3)

From (3), $\quad y_1''(x) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x \cos x.$ ...(4)

Now, $y_1''(x) - 2y_1'(x) + 2y_1(x) = 2e^x \cos x - 2e^x (\sin x + \cos x) + 2e^x \sin x = 0,$
showing that $y_1(x) = e^x \sin x$ is a solution of (1).

Similarly, we can show that $y_2(x) = e^x \cos x$ is a solution of (1).

Now, the Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x (\sin x + \cos x) & e^x (\cos x - \sin x) \end{vmatrix}$$

$$= e^{2x} (\sin x \cos x - \sin^2 x) - e^{2x} (\sin x \cos x + \cos^2 x) = -e^{2x} \neq 0,$$

showing that $W(x) \neq 0$, and hence $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (1).

The general solution of (1) is given by [Refer Art. 2.11]

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = e^x (c_1 \sin x + c_2 \cos x), \quad \ldots(5)$$

where $c_1$ and $c_2$ are arbitrary constants.

From (5), $y'(x) = e^x (c_1 \sin x + c_2 \cos x) + e^x (c_1 \cos x - c_2 \sin x)$....(6)

Putting $x = 0$ in (5) and using the given result $y(0) = 2$, we get

$$y(0) = c_2 \text{ or } c_2 = 2.$$

Putting $x = 0$ in (6) and using the given result $y'(0) = -3$, we get

$$y'(0) = c_2 + c_1 \text{ or } -3 = 2 + c_2 \text{ or } c_1 = -5.$$

$\therefore$ from (5), the solution of given equation satisfying the given properties is,

$$y = e^x (2 \cos x - 5 \sin x).$$

**Ex. 4.** *Show that $e^{2x}$ and $e^{3x}$ are linearly independent solutions of $y'' - 5y' + 6y = 0$. Find the solution $y(x)$ with the property that $y(0) = 0$ and $y'(0) = 1$.* **[Delhi B.Sc. (G) 1998]**

**Sol.** Given equation is $\quad y'' - 5y' + 6y = 0. \quad$ ...(1)

Let $\quad y_1(x) = e^{2x}$ and $y_2(x) = e^{3x}. \quad$ ...(2)

From (2) $\quad y_1'(x) = 2e^{2x}$ and $y_1''(x) = 4e^{2x}. \quad$ ...(3)

$$\therefore y_1''(x) - 5y_1'(x) + 6y_1(x) = 4e^{2x} - 5(2e^{2x}) + 6e^{2x} = 0,$$

showing that $y_1(x)$ is a solution of (1).

Similarly, we find that $y_2(x) = e^{3x}$ is a solution of (1),

Now, the Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x} = e^{5x} \neq 0,$$

showing that that $y_1(x) = e^{2x}$ and $y_2(x) = e^{3x}$ are linearly independent solutions of (1).

The general solution of (1) is given by

$y(x) = c_1 e^{2x} + c_2 e^{3x}$, $c_1$ and $c_2$ being arbitrary constants. $\quad$ ...(4)

From (4), $\quad y'(x) = 2c_1 e^{2x} + 3c_2 e^{3x}. \quad$ ...(5)

Putting $x = 0$ in (4) and using $y(0) = 0$, we get $c_1 + c_2 = 0$. ...(6)

Putting $x = 0$ in (5) and using $y'(0) = 1$, we get $2c_1 + 3c_2 = 1$. ...(7)

Solving (6) and (7), $c_1 = -1$ and $c_2 = 1$ and so from (4), we have

$$y(x) = e^{3x} - e^{2x} \text{ as the required solution.}$$

**Ex. 5.** *Show that $y_1(x) = \sin x$ and $y_2(x) = \sin x - \cos x$ are linearly independent solutions of $y'' + y = 0$. Determine the constants $c_1$ and $c_2$ so that the solution $\sin x + 3\cos x \equiv c_1 y_1(x) + c_2 y_2(x)$.* **[Delhi B.A. (P) 2002]**

**Sol.** Given equation is $\quad y'' + y = 0.$ ...(1)

Here $y_1(x) = \sin x$ so that $y_1'(x) = \cos x$ and $y_1''(x) = -\sin x$. ...(2)

Hence $\quad y_1''(x) + y_1(x) = -\sin x + \sin x = 0$, showing that $y_1(x)$ is a solution of (1). Similarly, we can show that $y_2(x)$ is also a solution of (1).

Now, the Wronskian of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin x & \sin x - \cos x \\ \cos x & \cos x + \sin x \end{vmatrix}$$

$$= \sin x\,(\cos x + \sin x) - \cos x\,(\sin x - \cos x) = 1 \neq 0,$$

showing that $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (1).

Given that $\sin x + 3\cos x \equiv c_1 y_1(x) + c_2 y_2(x)$

or $\quad \sin x + 3\cos x \equiv c_1 \sin x + c_2 (\sin x - \cos x)$. ...(3)

Comparing the coefficients of $\sin x$ and $\cos x$ on both sides of (3), we get

$$c_1 + c_2 = 1 \text{ and } -c_2 = 3 \text{ so that } c_1 = 4 \text{ and } c_2 = -3.$$

**Ex. 6.** *Show that $x$ and $x\,e^x$ are linearly independent on the x-axis.*

**Sol.** The Wronskian $W(x)$ of $x$ and $x\,e^x$ is given by

$$W(x) = \begin{vmatrix} x & xe^x \\ \dfrac{dx}{dx} & \dfrac{d(x\,e^x)}{dx} \end{vmatrix} = \begin{vmatrix} x & x\,e^x \\ 1 & e^x + x\,e^x \end{vmatrix}$$

$$= x\,(e^x + x\,e^x) - x\,e^x = x^2\,e^x.$$

We observe that $W(x) \neq 0$ for $x \neq 0$ on the $x$-axis. Hence $x$ and $x\,e^x$ are linearly independent on the $x$-axis [Refer corollary to theorem III of Art 2.6]

**Ex. 7.** *Show that the Wronskian of the functions $x^2$ and $x^2 \log x$ is non-zero. Can these functions be indpendent solutions of an ordinary differential equation. If so, determine this differential equation.* **[Meerut 1988, 98]**

**Sol.** Let $\quad y_1(x) = x^2$ and $y_2(x) = x^2 \log x$.

The Wronskian $W(x)$ of $y_1$ and $y_2$ is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2 \log x \\ 2x & 2x\log x + x \end{vmatrix} = x^2\,(2x\log x + x) - 2x^3 \log x.$$

$\therefore \quad W(x) = x^3$, which is not identically equal to zero on $(-\infty, \infty)$. Hence solution $y_1$ and $y_2$ can be linearly independent solutions of an ordinary differential equation.

**To form the required differential equation.** The general solution of the required differential equation may be written as

$$y = Ay_1 + By_2 = Ax^2 + Bx^2 \log x. \qquad ...(1)$$

where $A$ and $B$ are arbitrary constants.

Differentiating (1), $y' = 2Ax + B(2x \log x + x)$. ...(2)

Differentiating (2), $y'' = 2A + B(2 \log x + 2 + 1)$. ...(3)

We now eliminate $A$ and $B$ from (1), (2) and (3). To this end, we first solve (2) and (3) for $A$ and $B$. Multiplying both sides of (3) by $x$, we get

$$xy'' = 2Ax + B(3x + 2x \log x). \qquad ...(4)$$

Subtracting (2) from (4), $xy'' - y' = 2Bx$ or $B = (xy'' - y')/2x$.

Substituting this value of $B$ in (3), we have

$$2A = y'' - (1/2x)(xy'' - y')(3 + 2 \log x)$$

or $$A = (1/4x)[2xy'' - (xy'' - y')(3 + 2 \log x)].$$

Substituting the above values $A$ and $B$ in (1), we have

$$y = (x/4)[2xy'' - 3xy'' + 3y' - 2xy'' \log x + 2y' \log x] + (x/2) \log x (xy'' - y')$$

or $$4y = x(-xy'' + 3y' - 2xy'' \log x + 2y' \log x) + 2x \log x (xy'' - y')$$

or $$x^2y'' - 3xy' + 4y = 0,$$ which is the required equation.

**Ex. 8.** *Evaluate the Wronskian of the functions $x$ and $x e^x$. Hence conclude whether or not these are linearly independent. If they are independent, set up the differential equation having them as its independent solutions.*

[Meerut 97]

**Sol.** Let $y_1 = x$ and $y_2 = x e^x$. Then their Wronskian $W(x)$ is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = xe^x + x^2e^x - xe^x = x^2 e^x,$$

which is not identically equal to zero on $(-\infty, \infty)$. Hence $y_1$ and $y_2$ are linearly independent.

**To form the required differential equation.** The general solution of the required differential equation may be written as

$$y = Ay_1 + By_2 = Ax + Bx e^x, \qquad ...(1)$$

where $A$ and $B$ are arbitrary constants.

Differentiating (1), $y' = A + B(e^x + xe^x) = A + B(1 + x)e^x$. ...(2)

Differentiating (2), $y'' = B[e^x + (1 + x)e^x] = Be^x(2 + x)$. ...(3)

We now eliminate $A$ and $B$ from (1), (2) and (3). To this we first solve (2) and (3) for $A$ and $B$.

From (3), $B = y''/[e^x(2 + x)]$.

Substituting this value of $B$ in (2), we have

$$A = y' - B(1 + x)e^x = y' - \frac{1+x}{2+x} y'' = \frac{(2+x)y' - (1+x)y''}{2+x}.$$

Substituting the above values of $A$ and $B$ in (1), we get

$$y = \left[\frac{(2+x)y' - (1+x)y''}{2+x}\right]x + \left[\frac{y''}{e^x(2+x)}\right]x e^x$$

or $\qquad (2+x)y = x(2+x)y' - x(1+x)y'' + xy''$

or $\quad x^2y'' - x(2+x)y' + (2+x)y = 0$, which is required equation.

**Ex. 9.(a)** *Show that the solutions* $e^x, e^{-x}, e^{2x}$ of $(d^3y/dx^3) - 2(d^2y/dx^2) - (dy/dx) + 2y = 0$ *are linearly independent and hence or otherwise solve the given equation.* **[Delhi B.Sc. (G) 1993, 98 ; Meerut 87, 98]**

**Sol.** Given equation is $\qquad y''' - 2y'' - y' + 2y = 0.$ ...(1)

Let $\qquad y_1 = e^x, y_2 = e^{-x}$ and $y_3 = e^{2x}$. ...(2)

Here $\qquad y_1' = e^x, y_1'' = e^x$ and $y_1''' = e^x$. ...(3)

$\therefore \quad y_1''' - 2y_1'' - y_1' + 2y_1 = e^x - 2e^x - e^x + 2e^x = 0$, by (2) and (3)

Hence $\quad y_1 = e^x$ in a solution of (1). Similarly, we can show that $e^{-x}$ and $e^{2x}$ are also solutions of (1).

Now, the Wronskian $W(x)$ of $y_1, y_2, y_3$ in given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix}$$

$$= (e^x \cdot e^{-x} \cdot e^{2x}) \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = e^{2x} \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 1 \\ 1 & 0 & 3 \end{vmatrix} \text{ by } \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$= -6e^{2x}$, which is not identically zero on $(-\infty, \infty)$

Hence $y_1, y_2, y_3$ are linearly independent solutions of (1) [Refer corollary of theorem III of Art 2.6]. Since the order of the given equation (1) is three, it follows that the general solution of (1) will contain three arbitrary constants $c_1, c_2, c_3$ and is given by [Refer Art. 2.11]

$$y = c_1y_1 + c_2y_2 + c_3y_3 \text{ i.e., } y = c_1e^x + c_2e^{-x} + c_3e^{2x}.$$

**Ex. 9. (b)** *Show that the solutions* $e^x, e^{2x}, e^{-2x}$ of $y''' - y'' - 4y' + 4 = 0$ *are linearly independent and hence or otherwise solve the given equation.*

**Hint.** Try yourself as in Ex. 9. (a) **Ans.** $y = c_1e^x + c_2e^{2x} + c_3e^{-2x}$.

**Ex. 10.** *Prove that the functions* $1, x, x^2$ *are llinearly independent. Hence from the differential equation whose roots are* $1, x, x^2$. **[Meerut 1996, 97]**

**Sol.** Let $\qquad y_1(x) = 1, y_2(x) = x$ and $y_3(x) = x^2$. ...(1)

Then the Wronskian $W(x)$ of $y_1, y_2, y_3$ is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}, \text{ using (1)}$$

or $\qquad W(x) = 2 \neq 0$ for any $x \in (-\infty, \infty)$.

Hence, $y_1, y_2$, and $y_3$ are linearly independent.

2. Test the linear independence of the following sets of functions :

(i) $\sin x, \cos x$. [Ans. Linearly independent]

(ii) $1+x,\ 1+2x,\ x^2$. [Ans. Linearly independent]

(iii) $x^2-1,\ x^2-x+1,\ 3x^2-x-1$. [Ans. Linearly dependent]

(iv) $\sin x, \cos x, \sin 2x$. [Ans. Linearly independent]

(v) $e^x, e^{-x}, \sin ax$. [Ans. Linearly independent]

(vi) $e^x, x e^x, \sinh x$. [Ans. Linearly independent]

(vii) $\sin 3x, \sin x, \sin^3 x$. [Ans. Linearly dependent]

3. Show that the functions $e^x \cos x$ and $e^x \sin x$ are linearly independent. Form the differential equation of second order having these two functions as independent solutions. [Ans. $y''-2y'+2y=0$]

4. Evaluate the Wronskian of the functions $e^x$ and $x e^x$. Hence conclude whether or not they are linearly independent. If they are independent set up the differential equation having them as its independent solutions. [Ans. $y''-2y'+y=0$]

5. Show that any two solutions of the equation $y''+f(x)y'+g(x)y=0$, $f(x)$ and $g(x)$ being continuous on an open interval $I$, are linearly independent, if and only if, their Wronskian is zero for some $x=x_0$ on $I$. [Meerut 1992]

[Hint. Proceed exactly as in theorem V of Art. 2.13 for $n=2$.]

6. If the functions $p(x)$ and $q(x)$ are continuous on $\alpha<x<\beta$, and if the functions $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the differential equation $y''+p(x)y'+q(x)y=0$, then prove that the Wronskian $W(y_1, y_2)$ in non- vanishing on $\alpha<x<\beta$.

[Hint. Proceed exactly as in theorem V of Art 2.13 for $n=2$]

7. Show that linearly independent solutions of $y''-3y'+2y=0$ are $e^x$ and $e^{2x}$. Find the solution $y(x)$ with the property that $y(0)=0$, $y'(0)=1$. [Ans. $y(x)=e^{2x}-e^x$]

(Delhi B.Sc. (G) 2000)

8. Show that the $y_1(x)=x$ and $y_2(x)=|x|$ are linearly independent on the real line, even though the Wronskian cannot be computed.

9. Show graphically that $y_1(x)=x^2$ and $y_2(x)=x\,|\,x\,|$ are linearly independent on $-\infty<x<\infty$, however Wronskian vanishes for every real value of $x$.

10. Show that $e^x$ and $e^{-x}$ are linearly independent solutions of $y''-y=0$ on any interval. [Nagpur 96]

11. Show that $y_1(x)=e^{-x/2}\sin(x\sqrt{3}/2)$ and $y_2(x)=e^{-x/2}\cos(x\sqrt{3}/2)$ are lineraly independent solutions of the differential equation $y''+y'+y=0$. (Delhi B.Sc. (G) 1999, 2001)

[Hint: Proceed as in solved Ex. 2 on page 35]

# IAS Previous Years Questions (1983–2012) Segment-wise

## Ordinary Differential Equations and Laplace Transforms

### 1983

- Solve $x\frac{d^2y}{dx^2}+(x-1)\frac{dy}{dx}-y=x^2$.
- Solve $(y^2+yz)\,dx+(xz+z^2)\,dy+(y^2-xy)\,dz=0$.
- Solve the equation $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=t$ by the method of Laplace transform, given that $y=-3$ when $t=0$, $y=-1$ when $t=1$.

### 1984

- Solve $\frac{d^2y}{dx^2}+y=\sec x$.
- Using the transformation $y=\frac{u}{x^k}$ solve the equation $xy''+(1+2k)\,y'+xy=0$.
- Solve the equation $(D^2+1)x=t\cos 2t$, given that $x_0=x_1=0$, by the method of Laplace transform.

### 1985

- Consider the equation $y'+5y=2$. Find that solution $\phi$ of the equation which satisfies $\phi(1)=3\phi'(0)$.
- Use Laplace transform to solve the differential equation $x''-2x'+x=e^t,\left('=\frac{d}{dt}\right)$ such that $x(0)=2, x'(0)=-1$.
- For two functions f, g both absolutely integrable on $(-\infty,\infty)$, define the convolution $f*g$. If $L(f)$, $L(g)$ are the Laplace transforms of f, g show that $L(f*g)=L(f)\,L(g)$.
- Find the Laplace transform of the function
$$f(t)=\begin{cases}1 & 2n\pi\le t<(2n+1)\pi\\ -1 & (2n+1)\pi\le t\le(2n+2)\pi\end{cases}$$
$n=0,1,2,\ldots\ldots\ldots\ldots$

### 1987

- Solve the equation $x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}=y+e^x$
- If $f(t)=t^{p-1}$, $g(t)=t^{q-1}$ for $t>0$ but $f(t)=g(t)=0$ for $t\le 0$, and $h(t)=f*g$, the convolution of f, g show that $h(t)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}t^{p+q-1};t\ge 0$ and p, q are positive constants. Hence deduce the formula $B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

### 1988

- Solve the differential equation $\frac{d^2y}{dx^2}-2\frac{dy}{dx}=2e^x\sin x$.
- Show that the equation $(12x+7y+1)\,dx+(7x+4y+1)\,dy=0$ represents a family of curves having as asymptotes the lines $3x+2y-1=0$, $2x+y+1=0$.
- Obtain the differential equation of all circles in a plane in the form $\frac{d^3y}{dx^3}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}-3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2=0$.

### 1989

- Find the value of y which satisfies the equation $(xy^3-y^3-x^2e^x)+3xy^2\frac{dy}{dx}=0$; given that $y=1$ when $x=1$.
- Prove that the differential equation of all parabolas lying in a plane is $\frac{d^2}{dx^2}\left(\frac{d^2y}{dx^2}\right)^{-2/3}=0$.
- Solve the differential equation $\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}-6\frac{dy}{dx}=1+x^2$.

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## 1990

If the equation $\lambda^n + a_1\lambda^{n-1} + \dots\dots + a_n = 0$ (in unknown $\lambda$) has distinct roots $\lambda_1, \lambda_2, \dots\dots\dots \lambda_n$. Show that the constant coefficients of differential equation

$$\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \dots\dots\dots + a_{n-1}\frac{dy}{dx} + a_n = b$$ has the most general solution of the form

$y = c_0(x) + c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \dots\dots\dots\dots + c_n e^{\lambda_n x}$.

where $c_1, c_2, \dots\dots c_n$ are parameters. what is $c_0(x)$?

Analyse the situation where the $\lambda$ – equation in (a) has repeated roots.

Solve the differential equation

$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0$ is explicit form. If your answer contains imaginary quantities, recast it in a form free of those.

Show that if the function $\frac{1}{t - f(t)}$ can be integrated (w.r.t 't'), then one can solve $\frac{dy}{dx} = f(y/x)$ for any given $f$. Hence or otherwise.

$$\frac{dy}{dx} + \frac{x + 3y + 2}{3x + y + 6} = 0$$

- Verify that $y = (\sin^{-1}x)^2$ is a solution of $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$. Find also the most general solution.

## 1991

- If the equation $Mdx + Ndy = 0$ is of the form $f_1(xy)$. $ydx + f_2(xy).x\,dy = 0$, then show that $\frac{1}{Mx - Ny}$ is an integrating factor provided $Mx - Ny \neq 0$.
- Solve the differential equation $(x^2 - 2x + 2y^2)\,dx + 2xy\,dy = 0$.
- Given that the differential equation $(2x^2y^2 + y)\,dx - (x^3y - 3x)\,dy = 0$ has an integrating factor of the form $x^h y^k$, find its general solution.
- Solve $\frac{d^4y}{dx^4} - m^4 y = \sin mx$.
- Solve the differential equation

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = e^x$$

- Solve the differential equation

$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 5y = xe^{-x}$, given that $y = 0$ and $\frac{dy}{dx} = 0$, when $x = 0$.

## 1992

By eliminating the constants a, b obtain the differential equation of which $xy = ae^x + be^{-x} + x^2$ is a solution.

- Find the orthogonal trajectories of the family of semicubical parabolas $ay^2 = x^3$, where a is a variable parameter.
- Show that $(4x+3y+1)\,dx + (3x+2y+1)\,dy = 0$ represents hyperbolas having the following lines as asymptotes $x + y = 0$, $2x + y + 1 = 0$. **(1998)**
- Solve the following differential equation $y(1+xy)\,dx + x(1-xy)\,dy = 0$.
- Solve the following differential equation $(D^2+4)\,y = \sin 2x$ given that when $x = 0$ then $y = 0$ and $\frac{dy}{dx} = 2$.
- Solve $(D^3-1)y = xe^x + \cos^2 x$.
- Solve $(x^2D^2 + xD - 4)\,y = x^2$.

## 1993

- Show that the system of confocal conics

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$ is self orthogonal.

- Solve $\left\{y\left(1+\frac{1}{x}\right) + \cos y\right\}dx + \{x + \log x - x\sin y\}\,dy = 0$.

Solve $\frac{d^2y}{dx^2} + w_0^2 y = a\cos wt$ and discuss the nature of solution as $w \to w_0$.

- Solve $(D^4+D^2+1)\,y = e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right)$.

## 1994

- Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$.
- Show that if $\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is a function of x only say, f(x), then $F(x) = e^{\int f(x)dx}$ is an integrating factor of Pdx + Qdy = 0.
- Find the family of curves whose tangent form angle $\pi/4$ with the hyperbola xy = c.
- Transform the differential equation $\frac{d^2y}{dx^2}\cos x + \frac{dy}{dx}\sin x - 2y\cos^3 x = 2\cos^5 x$ into one having z an independent variable where z = sin x and solve it.
- If $\frac{d^2x}{dt^2} + \frac{g}{b}(x-a) = 0$, (a, b and g being positive constants) and x = a′ and $\frac{dx}{dt} = 0$ when t=0, show that $x = a + (a'-a)\cos\sqrt{\frac{g}{b}}t$.
- Solve $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x$, where, $D = \frac{d}{dx}$.

## 1995

- Solve $(2x^2+3y^2-7)x\,dx - (3x^2+2y^2-8)\,y\,dy = 0$.
- Test whether the equation $(x+y)^2 dx - (y^2-2xy-x^2) dy = 0$ is exact and hence solve it.
- Solve $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x+\frac{1}{x}\right)$. **(1998)**
- Determine all real valued solutions of the equation $y''' - iy'' + y' - iy = 0$, $y' = \frac{dy}{dx}$
- Find the solution of the equation $y'' + 4y = 8\cos 2x$ given that $y = 0$ and $y' = 2$ when x = 0.

## 1996

- Solve $x^2(y-px) = yp^2$, $\left(p = \frac{dy}{dx}\right)$
- Solve $y \sin 2x\,dx - (1+y^2+\cos^2 x)\,dy = 0$.
- Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y + 37\sin 3x = 0$. Find the value of y when $x = \pi/2$, if it is given that y=3 and $\frac{dy}{dx} = 0$ when x=0.
- Solve $\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} = x^2 + 3e^{2x} + 4\sin x$.
- Solve $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x + \log x$.

## 1997

- Solve the initial value problem $\frac{dy}{dx} = \frac{x}{x^2y + y^3}$, y(0)=0.
- Solve $(x^2-y^2+3x-y)dx + (x^2-y^2+x-3y)dy = 0$.
- Solve $\frac{d^4y}{dx^4} + 6\frac{d^3y}{dx^3} + 11\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 20e^{-2x}\sin x$
- Make use of the transformation y(x) = u(x) sec x to obtain the solution of $y'' - 2y'\tan x + 5y = 0$; y(0)=0; $y'(0) = \sqrt{6}$.
- Solve $(1+2x)^2\frac{d^2y}{dx^2} - 6(1+2x)\frac{dy}{dx} + 16y = 8(1+2x)^2$; $y(0) = 0$ and $y'(0) = 2$.

## 1998

- Solve the differential equation $xy - \frac{dy}{dx} = y^3e^{-x^2}$
- Show that the equation (4x+3y+1) dx + (3x+2y+1) dy = 0 represents a family of hyperbolas having as asymptotes the lines x+y = 0; 2x+y+1=0. (1992)
- Solve the differential equation $y = 3px + 4p^2$.
- Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}(x^2+9)$.

- Solve the differential equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=x\sin x.$$

## 1999

- Solve the differential equation

$$\frac{xdx+ydy}{xdy-ydx}=\left(\frac{1-x^2-y^2}{x^2+y^2}\right)^{1/2}$$

- Solve $\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+4\frac{dy}{dx}-2y=e^x+\cos x.$

- By the method of variation of parameters solve the differential equation $\frac{d^2y}{dx^2}+a^2y=\sec(ax)$.

## 2000

- Show that $3\frac{d^2y}{dx^2}+4x\frac{dy}{dx}-8y=0$ has an integral which is a polynomial in x. Deduce the general solution. 12

- Reduce $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$, where P, Q, R are functions of x to the normal form.

  Hence solve $\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(4x^2-1)y=-3e^{x^2}\sin 2x$. 15

- Solve the differential equation $y=x-2ap+ap^2$. Find the singular solution and interpret it geometrically. 15

- Show that $(4x+3y+1)dx+(3x+2y+1)dy=0$ represents a family of hyperbolas with a common axis and tangent at the vertex. 15

- Solve $x\frac{dy}{dx}-y=(x-1)\left(\frac{d^2y}{dx^2}-x+1\right)$ by the method of variation of parameters. 15

## 2001

- A continuous function y(t) satisfies the differential equation

$$\frac{dy}{dt}=\begin{cases}1+e^{-t}, & 0\le t<1\\ 2+2t-3t^2, & 1\le t\le 5\end{cases}$$

If y(0) = –e, find y(2). 12

- Solve $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}-3y=x^2\log x$ 12

- Solve $\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y(\log y)^2}{x^2}$ 15

- Find the general solution of $ayp^2+(2x-b)p-y=0$, a>o. 15

- Solve $(D^2+1)^2y=24x\cos x$

  given that $y=Dy=D^2y=0$ and $D^3y=12$ when $x=0$. 15

- Using the method of variation of parameters, solve

$$\frac{d^2y}{dx^2}+4y=4\tan 2x$$ 15

## 2002

- Solve $x\frac{dy}{dx}+3y=x^3y^2$ 12

- Find the values of λ for which all solutions of $x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}-\lambda y=0$ tend to zero as $x\to\infty$. 12

- Find the value of constant λ such that the following differential equation becomes exact.

$$\left(2xe^y+3y^2\right)\frac{dy}{dx}+\left(3x^2+\lambda e^y\right)=0$$

Further, for this value of λ, solve the equation. 15

- Solve $\frac{dy}{dx}=\frac{x+y+4}{x-y-6}$. 15

- Using the method of variation of parameters, find the solution of $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=xe^x\sin x$ with $y(0)=0$ and $\left(\frac{dy}{dx}\right)_{x=0}=0$. 15

- Solve $(D-1)(D^2-2D+2)y=e^x$ where $D=\frac{d}{dx}$. 15

## 2003

- Show that the orthogonal trajectory of a system of confocal ellipses is self orthogonal. 12
- Solve $x\frac{dy}{dx}+y\log y = xye^x$. 12
- Solve $(D^3-D)\,y = 4(e^x+\cos x + x^3)$, where $D=\frac{d}{dx}$. 15
- Solve the differential equation $(px^2+y^2)(px+y) = (p+1)^2$ where $p=\frac{dy}{dx}$, by reducing it to Clairaut's form using suitable substitutions. 15
- Solve $(1+x)^2y''+(1+x)y'+y=\sin 2\left[\log(1+x)\right]$. 15
- Solve the differential equation $x^2y''-4xy'+6y=x^4\sec^2 x$ by variation of parameters. 15

## 2004

- Find the solution of the following differential equation $\frac{dy}{dx}+y\cos x=\frac{1}{2}\sin 2x$. 12
- Solve $y(xy+2x^2y^2)\,dx + x(xy-x^2y^2)\,dy=0$. 12
- Solve $\left(D^4-4D^2-5\right)y=e^x(x+\cos x)$. 15
- Reduce the equation $(px-y)(py+x)=2p$ where $p=\frac{dy}{dx}$ to Clairaut's equation and hence solve it. 15
- Solve $(x+2)\frac{d^2y}{dx^2}-(2x+5)\frac{dy}{dx}+2y=(x+1)e^x$. 15
- Solve the following differential equation $(1-x^2)\frac{d^2y}{dx^2}-4x\frac{dy}{dx}-(1+x^2)y=x$. 15

## 2005

- Find the orthogonal trajectory of a system of co-axial circles $x^2+y^2+2gx+c=0$, where g is the parameter. 12
- Solve $xy\frac{dy}{dx}=\sqrt{x^2-y^2-x^2y^2-1}$ 12
- Solve the differential equation $(x+1)^4D^3+2(x+1)^3D^2-(x+1)^2D+(x+1)y=\frac{1}{x+1}$ 15
- Solve the differential equation $(x^2+y^2)(1+p)^2-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$, where $p=\frac{dy}{dx}$, by reducing it to Clairaut's form by using suitable substitution. 15
- Solve the differential equation $(\sin x - x\cos x)\,y''-x\sin x\,y'+y\sin x=0$ given that $y=\sin x$ is a solution of this equation. 15
- Solve the differential equation $x^2y''-2xy'+2y=x\log x, x>0$ by variation of parameters. 15

## 2006

- Find the family of curves whose tangents form an angle $\pi/4$ with the hyperbolas $xy=c$, $c>0$. 12
- Solve the differential eqaution $\left(xy^2+e^{-1/x^3}\right)dx-x^2y\,dy=0$. 12
- Solve $(1+y^2)+\left(x-e^{-\tan^{-1}y}\right)\frac{dy}{dx}=0$. 15
- Solve the equation $x^2p^2+yp(2x+y)+y^2=0$ using the substitution $y=u$ and $xy=v$ and find its singular solution, where $p=\frac{dy}{dx}$. 15
- Solve the differential equation $x^2\frac{d^3y}{dx^3}+2x\frac{d^2y}{dx^2}+2\frac{y}{x}=10\left(1+\frac{1}{x^2}\right)$ 15
- Solve the differential equation $\left(D^2-2D+2\right)y=e^x\tan x$, where $D=\frac{d}{dx}$, by the method of variation of parameters. 15

## 2007

- Solve the ordinary differential equation

$$\cos 3x \frac{dy}{dx} - 3y \sin 3x = \frac{1}{2}\sin 6x + \sin^2 3x, \ 0 < x < \frac{\pi}{2}. \qquad 12$$

- Find the solution of the equation

$$\frac{dy}{y} + xy^2 dx = -4x dx \qquad 12$$

- Determine the general and singular solutions of the equation $y = x\frac{dy}{dx} + a\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-1/2}$ 'a' being a constant. 15

- Obtain the general solution of $\left[D^3 - 6D^2 + 12D - 8\right]$

$$y = 12\left(e^{2x} + \frac{9}{4}e^{-2x}\right), \text{ where } D = \frac{d}{dx}. \qquad 15$$

- Solve the equation $2x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 3y = x^3$. 15

- Use the method of variation of parameters to find the general solution of the equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2e^x \qquad 15$$

## 2008

- Solve the differential equation

$$y dx + \left(x + x^3y^2\right)dy = 0. \qquad 12$$

- Use the method of variation of parameters to find the general solution of $x^2y'' - 4xy' + 6y = -x^4 \sin x$. 12

- Using Laplace transform, solve the initial value problem $y'' - 3y' + 2y = 4t + e^{3t}$ with $y(0) = 1$, $y'(0) = -1$. 15

- Solve the differential equation

$$x^3y'' - 3x^2y' + xy = \sin(\ln x) + 1. \qquad 15$$

- Solve the equation $y - 2px + yp^2 = 0$ where $p = \frac{dy}{dx}$. 15

## 2009

- Find the Wronskian of the set of functions $\{3x^3, |3x^3|\}$ on the interval $[-1, 1]$ and determine whether the set is linearly dependent on $[-1, 1]$. 12

- Find the differential equation of the family of circles in the $xy$-plane passing through $(-1, 1)$ and $(1, 1)$. 20

- Fidn the inverse Laplace transform of

$$F(s) = \ln\left(\frac{s+1}{s+5}\right) \qquad 20$$

- Solve: $\frac{dy}{dx} = \frac{y^2(x-y)}{3xy^2 - x^2y - 4y^3}, y(0) = 1.$ 20

## 2010

- Consider the differential equation

$y' = \alpha x, x > 0$

where $\alpha$ is a constant. Show that–

(i) if $\phi(x)$ is any solution and $\Psi'(x) = \phi(x)e^{-\alpha x}$, then $\Psi(x)$ is a constant;

(ii) if $\alpha < 0$, then every solution tends to zero as $x \to \infty$. 12

- Show that the differntial equation

$(3y^2 - x) + 2y(y^2 - 3x)y' = 0$

admits an integrating factor which is a function of $(x+y^2)$. Hence solve the equation. 12

- Verify that

$$\frac{1}{2}(Mx+Ny)d\left(\log_e(xy)\right) + \frac{1}{2}(Mx-Ny)d\left(\log_e\left(\tfrac{x}{y}\right)\right)$$

$= M\,dx + N\,dy$

Hence show that–

(i) if the differential equation $M\,dx + N\,dy = 0$ is homogeneous, then $(Mx + Ny)$ is an integrating factor unless $Mx + Ny \equiv 0$;

(ii) if the differential equation

$Mdx + Ndy = 0$ is not exact but is of the form

$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$

then $(Mx - Ny)^{-1}$ is an integrating factor unless $Mx - Ny \equiv 0$. 20

- Show that the set of solutions of the homogeneous linear differential equation

$y' + p(x)y = 0$

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on an interval $I=[a,b]$ forms a vector subspace W of the real vector space of continuous functions on $I$. what is the dimension of W? 20

- Use the method of undetermined coefficients to find the particular solution of $y''+y=\sin x+(1+x^2)e^x$ and hence find its general solution. 20

**2011**

- Obtain the soluton of the ordinary differential equation $\frac{dy}{dx}=(4x+y+1)^2$, if y(0) = 1. 10
- Determine the orthogonal trajectory of a family of curves represented by the polar equation $r=a(1-\cos\theta)$, $(r,\theta)$ being the plane polar coordinates of any point. 10
- Obtain Clairaut's orm of the differential equation $\left(x\frac{dy}{dx}-y\right)\left(y\frac{dy}{dx}+x\right)=a^2\frac{dy}{dx}$. Also find its general solution. 15
- Obtain the general solution of the second order ordinary differential equation $y''-2y'+2y=x+e^x\cos x$, where dashes denote derivatives w.r.t. x. 15
- Using the method of variation of parameters, solve the second order differedifferential equation $\frac{d^2y}{dx^2}+4y=\tan 2x$ 15
- Use Laplace transform method to solve the following initial value problem: $\frac{d^2x}{dt^2}-2\frac{dx}{dt}+x=e^t$, $x(0)=2$ and $\left.\frac{dx}{dt}\right|_{t=0}=-1$ 15

**2012**

- Solve $\frac{dy}{dx}=\frac{2xye^{(x/y)^2}}{y^2(1+e^{(x/y)^2})+2x^2e^{(x/y)^2}}$ (12)
- Find the orthogonal trajectories of the family of curves $x^2+y^2=ax$. (12)
- Using Laplace transforms, solve the intial value problem $y''+2y'+y=e^{-t}$, $y(0)=-1$, $y'(0)=1$ (12)
- Show that the differential equation $(2xy\log y)dx+\left(x^2+y^2\sqrt{y^2+1}\right)dy=0$ is not exact. Find an integrating factor and hence, the solution of the equation. (20)
- Find the general solution of the equation $y'''-y''=12x^2+6x$. (20)
- Solve the ordinary differential equation $x(x-1)y''-(2x-1)y'+2y=x^2(2x-3)$ (20)

- Solve $\{x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1\}y = (1+\log x)^2$,

  Where $D = \frac{d}{dx}$. (15)

- Solve $(D^4 + D^2 + 1)y = ax^2 + be^{-x}\sin 2x$, where

  $D \equiv \frac{d}{dx}$ (15)

- Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$. (8)

- Solve and find the singular solution of

  $x^3p^2 + x^2py + a^3 = 0$ (8)

- Solve: $x^2y\frac{d^2y}{dx^2} + \left(x\frac{dy}{dx} - y\right)^2 = 0$ (10)

- Solve $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = x^2\cos x$. (10)

- Solve $x = y\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$ (10)

- Solve $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = (1$ [illegible] (10)



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# Previous Years Questions (2000–2012) Segment-wise

IFoS

## Ordinary Differential Equations

(According to the New Syllabus Pattern) Paper - I

### 2000

- Solve $(x^2+y^2)(1+p)^2-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$ $p=\frac{dy}{dx}$. Interpret geometrically the factors in the P-and C-discriminants of the equation $8p^3x=y(12p^2-9)$ (20)

- Solve

  (i) $\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+\frac{a^2}{x^4}y=0$

  (ii) $\frac{d^2y}{dx^2}+(\tan x-3\cos x)\frac{dy}{dx}+2y\cos^2 x=\cos^4 x$. by varying parameters. (20/2007)

### 2001

- A constant coefficient differential equation has auxiliary equtiion expressible in factored form as $P(m)=m^3(m-1)^2(m^2+2m+5)^2$. What is the order of the differential equation and find its general solution. (10)

- Solve $x^2\left(\frac{dy}{dx}\right)^2+y(2x+y)\frac{dy}{dx}+y^2=0$ (10)

- Using differential equations, show that the system of confocal conics given by $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$, is self othogonal. (10)

- Solve $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$ given that $y=e^{a\sin^{-1}x}$ is one solution of this equation. (10)

- Find a general solution $y''+y=\tan x$, $-\pi/2<x<\pi/2$ by variation of parameters. (10)

### 2002

- If $(D-a)^4e^{nx}$ is denoted by $z$, prove that $z, \frac{\partial z}{\partial n}, \frac{\partial^2 z}{\partial n^2}, \frac{\partial^3 z}{\partial n^3}$ all vanish when $n=a$. Hence show that $e^{nx}, xe^{nx}, x^2e^{nx}, x^3nx$ are all solutions of $(D-a)^4y=0$. Here $D$ Stands for $\frac{d}{dx}$. (10)

- Solve $4xp^2-(3x+1)^2=0$ and examine for singular solutions and extraneous loci. Interpret the results geometrically. (10)

- (i) Form the differential equation whose primitive is $y=A\left(\sin x+\frac{\cos x}{x}\right)+B\left(\cos x-\frac{\sin x}{x}\right)$

  (ii) Prove that the orthogonal trajectory of system of parabolas belongs to the system itself. (10)

- Using variation of parameters solve the differential equation $\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(4x^2-1)y=-3e^{x^2}\sin 2x$. (10)

- (i) Solve the equation by finding an integrating factor of $(x+2)\sin y\,dx+x\cos y\,dy=0$.

  (ii) Verify that $\phi(x)=x^2$ is a solution of $y''-\frac{2}{x^2}y=0$ and find a second independent solution. (10)

- Show that the solution of $(D^{2n+1}-1)y=0$, consists of $Ae^x$ and $n$ paris of terms of the form $e^{\alpha x}(b_r\cos\alpha x+c_r\sin\alpha x)$, Where $\alpha=\cos\frac{2\pi r}{2n+1}$ and $\alpha=\sin\frac{2\pi r}{2n+1}$; $r=1,2,\ldots,n$ and $b_r, c_r$ are arbitrary constants. (10)

### 2003

- Find the orthogonal trajectories of the family of co-axial circles $x^2+y^2+2gx+c=0$ Where $g$ is a parameter. (10)

- Find the three solutions of $\frac{d^3y}{dx^3}-2\frac{d^2y}{dx^2}-\frac{dy}{dx}+2y=0$ Which are linealy independent on every real interval. (10)

- Solve and examine for singular solution:

$$y^2-2pxy+p^2\left(x^2-1\right)=m^2.$$

(10)

- Solve $x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y=10\left(x+\frac{1}{x}\right)$ (10)

- Given $y=x$ is one solutions of $\left(x^3+1\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$ find another linearly independent solution by reducing order and write the general solution. (10)

- Solve by the method of variation of parameters $\frac{d^2y}{dx^2}+a^2y=\sec ax$, a real. (10)

**2004**

- Dertermine the family of orthgonal trajectories of the family $y=x-1+ce^{-x}$ (10)

- Show that the solution curve satisfying $\left(x^2-xy\right)y'=y$, Where $y\to 1$ as $x\to\infty$ is a conic section. Indentify the curve. (10)

- Solve $(1+x)^2y''+(1+x)y'+y=4\cos(\ln(1+x))$ , $y(0)=1, y(e-1)=\cos 1$. (10)

- Obtain the general solution of: $y''+2y'+2y=4e^{-x}x^2\sin x$. (10)

- Find the general solution of $\left(xy^3+y\right)dx+2\left(x^2y^2+x+y^4\right)dy=0$ (10)

- Obtain the general solution of $\left(D^4+2D^3-D^2-2D\right)y=x+e^{2x}$, Where $D=\frac{dy}{dx}$ (10)

**2005**

- Form the differential equation that represents all parabolas each of which has latus rectum $4a$ and Whose are parallel to the x-axis. (10)

- (i) The auxiliary polynomial of a certain homogenous linear differential equation with constant coefficients in factored form is $P(m)=m^4(m-2)^6\left(m^2-6m+25\right)^3$ What is the order of the differential equation and write a general solution ?
  (ii) Find the equation of the one-parameter family of parabolas given by $y^2=2cx+c^2$, $C$ real and show that this family is self-orthogonal. (10)

- Solve and examine for singular solution the following equation $P^2\left(x^2-a^2\right)-2pxy+y^2-b^2=0$ (10)

- Solve the differential equation $\frac{d^2y}{dx^2}+9y=\sec 3x$ (10)

- Given $y=x+\frac{1}{x}$ is one solution solve the differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$ reduction of order method (10)

- Find the general solution of the defferential equation $\frac{d^2y}{dx^2}-2\frac{dy}{dx}-3y=2e^x-10\sin x$ by the method of underternined coefficients. (10)

**2006**

- From $x^2+y^2+2ax+2by+c=0$, derive differential equation not containing, $a,b$ or $c$. (10)

- Discuss the solution of the differential equation

$$y^2=\left[1+\left(\frac{dy}{dx}\right)^2\right]=a^2$$

(10)

- Solve $x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}-y=e^x$ (10)

- Solve $\frac{d^4y}{dx^4}-y=x\sin x$ (10)

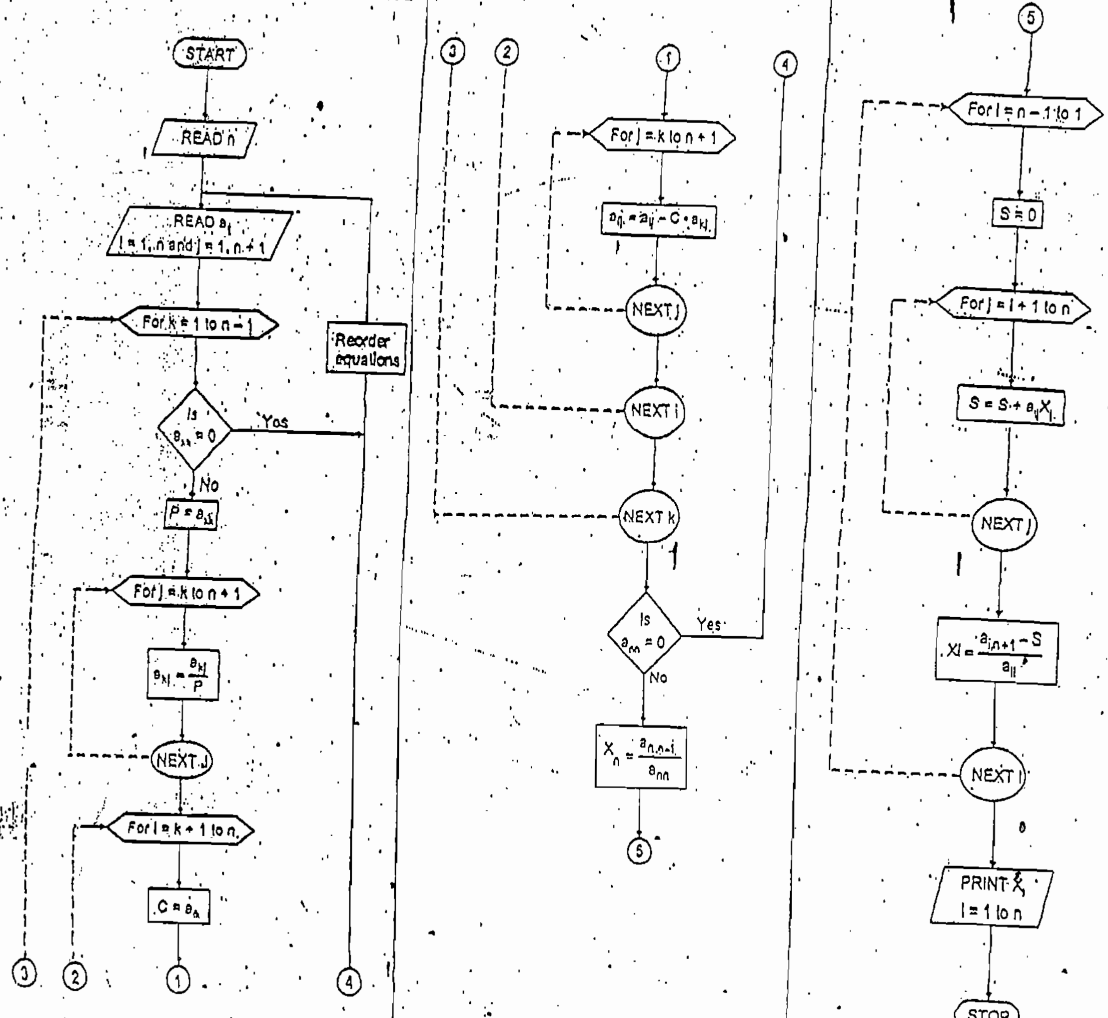
- Solve $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^2e^x$ (10/2008)

- Reduce

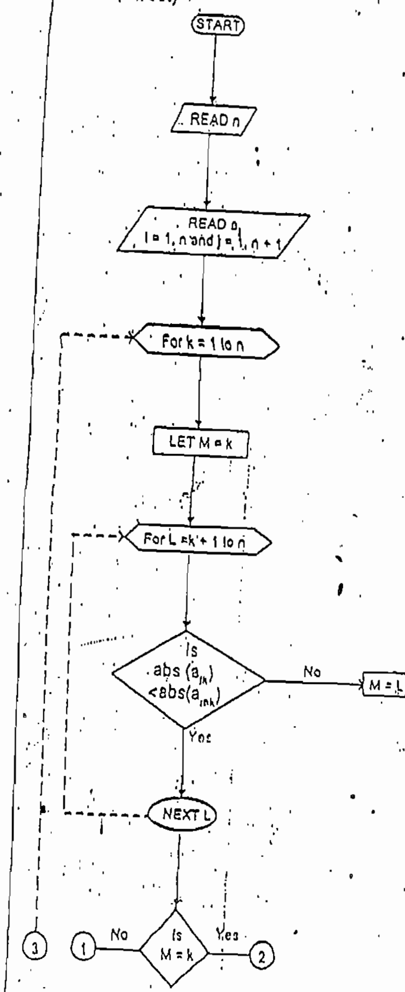
$$xy\left(\frac{dy}{dx}\right)^2-\left(x^2+y^2+1\right)\frac{dy}{dx}+xy=0$$

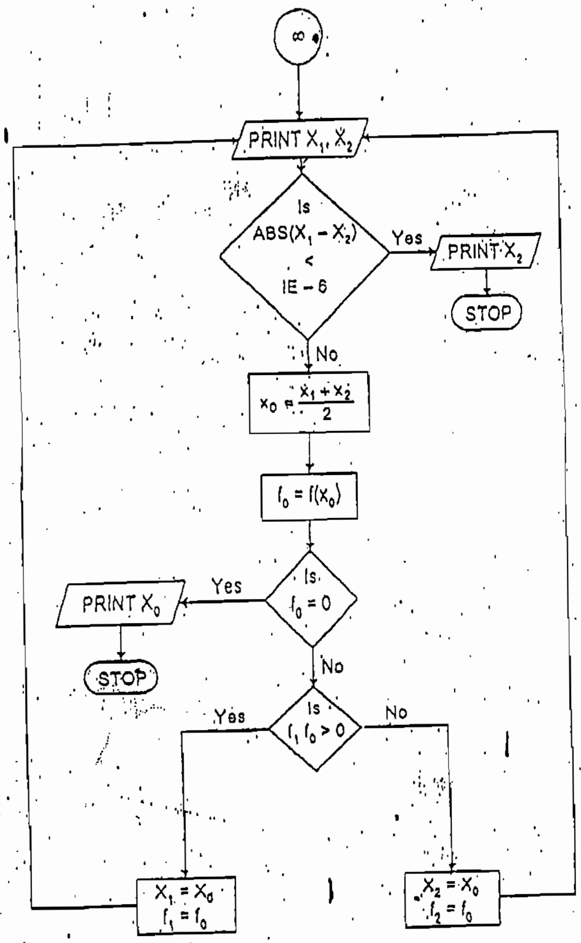
to clairaut's form and find its singular solution. (10)

(4) Flow chart for Gauss elimination method



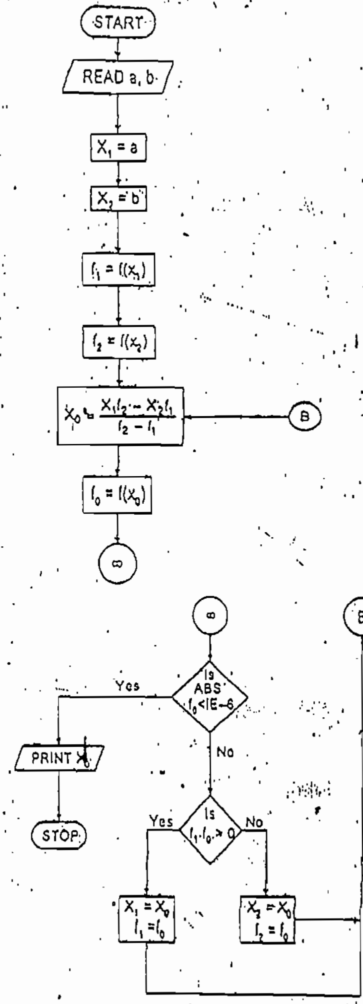

(5) Flow chart for Gauss-Jordan metho (Direct)



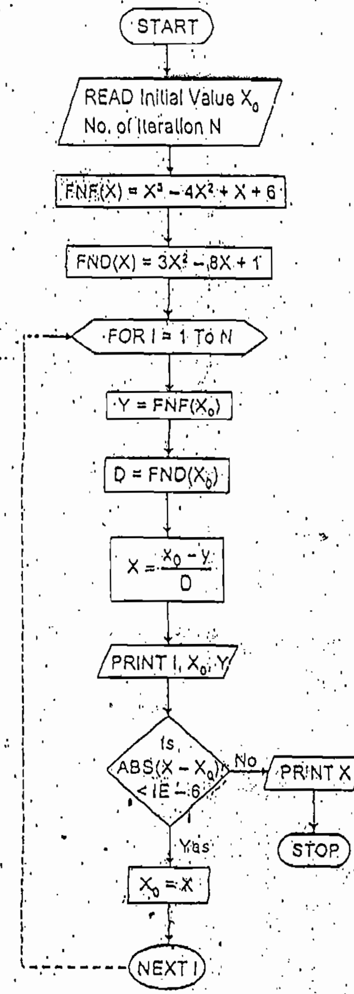





(2) Flow chart for Regula-Falsi method




(3) Flow chart for Newton-Raphson Method



(13) Euler's method

Step 1: Read initial values say a, b, y

Step 2: Read number of points of x (say n)

Step 3: Find the width of the interval say

$h = \frac{b-a}{n-1}$

Step 4: Assign a to x

Step 5: Assign I = 1

Step 6: If $x > = (b + h)$, follow Step 9; otherwise follow Step 7 and 8

Step 7: Evaluate the Eulers formula $y_{i+1} = y_i + hf(x_i, y_i)$ and print the results

Step 8: Increment I, I = I+1, follow step 6.

Step 9: Stop

(14) Runge – Kutta method

Step 1: Read initial values say a, b, y

Step 2: Read number of points of x (say n)

Step 3: Find the width of the interval (say h)

Step 4: Assign the initial value 'a' to 'x'

Step 5: Compute the value of the parameters viz., $k_1, k_2, k_3, k_4$ and y and substitute in the Runge-Kutta fourth order formula as follows:

$y = y + 1/6(k_1 + 2k_2 + 2k_3 + k_4)$

Step 6: Increment x, say x = x + h

Step 7: Continue steps 5 and 6 till we get number of points equal to n.

Step 8: Stop

## 5.FLOW CHARTING

### 6.0 Introduction

A flow chart is a pictorial representation of an algorithm in which boxes of different shapes are used to denote different types of operation.

The actual operation are stated within the boxes. The boxes are connected by directed solid lines indicating the flow of operations. Usually a flow chart is drawn before writing the programs and the flow chart is expressed in the programming language to prepare a program. The main advantage of this two-tier system in program writing is that while drawing a flow chart one is not concerned with the details of the elements of a programming language, they concentrate only on logic of the procedure. Further, the flow chart shows the logic pictorially, any problem can be identified and eliminated immediately.

It is important to note that, for a beginner, it is recommended that a flow chart be drawn first in order to reduce the number of errors and omissions in the program. It is a good practice to have a flow chart which may help during the testing of the program as well as while incorporating further modifications in the program.

### 5.1 Classification of flow charts

Flow charts can be divided into two broad categories:

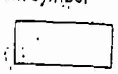
(1) Program flow charts and

(2) System flow charts

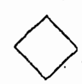
Program flow chart is the pictorial representation of a sequence of instructions for solving a problem. System flow chart indicates the flow of data throughout a data processing system, as well as the flow into and out of the system.

### 5.2 Flow chart symbols

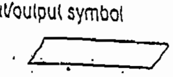
(1) Assignment symbol



(2) Decision symbol

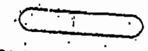


(3) Input/output symbol



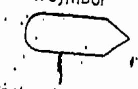


(4) Terminal symbol

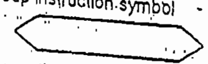


(5) Off-page connector symbol

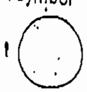
(6) Modification symbol



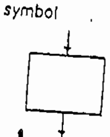
(7) Group instruction symbol



(8) Connection symbol



(9) Module symbol



### 5.3 Guidelines to draw efficient flow charts

(1) Analyse the nature of the problem completely and ensure that all the information relating to the problem is available

(2) Decide the sequence of actions to be taken so that solution can be obtained in a finite number of steps.

(3) Prepare the general algorithm

(4) Include refinements, wherever necessary

(5) Draw flow chart on the basis of the algorithm

(6) Draw flow chart symbols from top to bottom or left to right across the page to ensure the readability.

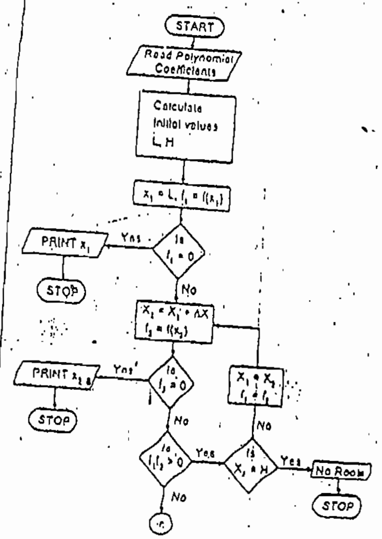
(7) Avoid intersecting directed lines. Use connector symbol wherever required.

(8) Always use pencil to draw flow chart so that erasing and redrawing are easier. To draw flow chart, use template.

(9) Use visible arrow head on directed lines whenever flow is not from top to bottom or from left to right.

### 5.4 Flow Charts for Solving Numerical Analysis Problems

(1) Flow chart for Bisection method



Step 4: Construct the difference table

Step 5: Compute the value of $u = \frac{a - a_0}{h}$

Step 6: Expand the Newton-forward formula and substitute the values to get the function value for the given argument (say a)

Step 7: Print the function value of the given argument (say a)

Step 8: If you want to continue, follow Steps 3 to 7, otherwise follow Step 9

Step 9: Stop

(8) Newton's – Backward formula

Step 1: Read n

Step 2: Read $x_i$ and $y_i$, i=1 to n

Step 3: Read the value of x (say a)

Step 4: Construct the difference table

Step 5: Compute the value of $u = \frac{a - a_0}{h}$

Step 6: Expand the Newton-forward formula and substitute the values, to get the function value for the given argument (say a)

Step 7: Print the function value of the given argument (say a)

Step 8: If you want to continue, follow Steps 3 to 7, otherwise follow Step 9

Step 9: Stop

(9) Lagrange's method

Step 1: Read the number of values (say n)

Step 2: Read $x_i$, i = 1 to n

Step 3: Read $y_i$, i = 1 to n

Step 4: Read the value of x (say a)

Step 5: Expand the lagrangian interpolation formula as per 'n'

Step 6: Substitute the values of a, $x_i$ and $y_i$ and evaluate.

Step 7: Print the results (i.e. x and y)

Step 8: If you want to try for another value follow Steps 4 to 8, otherwise follow Step 9

Step 9: Stop

(10) Simpson's One-third rule

Step 1: Read an initial value, say a

Step 2: Read the final value, say b

Step 3: Read number of divisions, say n

Step 4: Find the width of the interval

$$\text{i.e. } h = \frac{b-a}{n}$$

Step 5: Assign m=n/2 (to get the area under the curve for n intervals)

Step 6: Assign S = 0 and x = a

Step 7: Expand the Simpson's rule and is as follows

$$\frac{h}{3}(y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5$$

$$+ \ldots + 2y_{n-1} + 4y_n + y_{n+1})$$ and compute the values of h and $y_i$ and evaluate.

Step 8: Print the results.

Step 9: Stop

(11) Trapezoidal rule

Step 1: Read two end points of the interval say $x_1, x_2$

Step 2: Read the allowed error in the integral say e

Step 3: $h \longleftarrow x_2 - x_1$

Step 4: $S_1 \longleftarrow (f(x_1) + f(x_2))/2$

Step 5: $I_1 \longleftarrow h \times S_1$

Step 6: $S_0 \longleftarrow 0$

Step 7: $I_0 \longleftarrow 0$



Step 8: $i \longleftarrow 1$

Step 9: When $|(I_1 - I_0)/I_1| > e$ then follow Step 10 to 16, otherwise follow Step 17

Step 10: $S_0 \longleftarrow S_1$

Step 11: $x \longleftarrow x_1 + h/2$

Step 12: for j = 1 to i

$S_1 \longleftarrow S_1 + f(x)$

$x \longleftarrow x + h$

next j

Step 13: $i \longleftarrow 2 \times i$

Step 14: $h \longleftarrow h/2$

Step 15: $I_0 \longleftarrow I_1$

Step 16: $I_1 \longleftarrow h \times S_1$

Step 17: Print the results $I_1$, h

Step 18: Stop

(12) Gaussian Quadrature formula

Step 1: First transform the independent variable

Step 2: Choose value of n

Step 3: Expand the Gauss formula

Step 4: Substitute the values for the parameters $A_i$ and $Z_i$ (use the following Table) and evaluate the formula

Parameters for Gaussian Integration

| n | i | $A_i$ | $Z_i$ |
|---|---|---|---|
| 2 | 1 | 1.00000 | -0.57735 |
| | 2 | 1.00000 | 0.57735 |
| 3 | 1 | 0.55556 | -0.77460 |
| | 2 | 0.88889 | 0.00000 |
| | 3 | 0.55556 | -0.77460 |
| 4 | 1 | 0.34785 | -0.86114 |
| | 2 | 0.65215 | -0.33998 |
| | 3 | 0.65215 | 0.33998 |
| | 4 | 0.34785 | 0.86114 |
| 5 | 1 | 0.23693 | -0.90618 |
| | 2 | 0.47863 | -0.53847 |
| | 3 | 0.56889 | 0.00000 |
| | 4 | 0.47863 | 0.53847 |
| | 5 | 0.23693 | 0.90618 |
| 6 | 1 | 0.17132 | -0.93247 |
| | 2 | 0.36076 | -0.66121 |
| | 3 | 0.46791 | -0.23862 |
| | 4 | 0.46791 | 0.23862 |
| | 5 | 0.36076 | 0.66121 |
| | 6 | 0.17132 | 0.93247 |

### 4.2 Essential steps to be followed for development

From this, it can be stated that the following steps are very essential to develop an algorithm for any task.

(1) Input step

(2) Assignment step

(3) Decision step

(4) Repetitive step and

(5) Output step

Remark: The input step takes the three marks along with register number for each student. Assignment step assigns three marks and register number to SUB1, SUB2, SUB3 and REGNO. The decision step checks whether or all the students have been considered. The repetitive steps concentrate on calculation of average marks for all the 100 students. The output step provides the result.

### 4.3 Illustrative examples of algorithms using Numerical methods

(1) Bisection method

Step 1: Read initial values say $x_0$ and $x_1$

Step 2: Read the allowed relative error say e (epsilon)

Step 3: $y_0 \longleftarrow f(x_0)$

Step 4: $y_1 \longleftarrow f(x_1)$

Step 5: $i \longleftarrow 0$

Step 6: If $y_0 y_1 > 0$ follow steps 14 to 16; otherwise follow steps 7 to 14

Step 7: When $|(x_1 - x_0)/x_1| > e$, follow steps 8 to 11

Step 8: $x_2 \longleftarrow (x_0 + x_1)/2$

Step 9: $y_2 \longleftarrow f(x_2)$.

Step 10: $i \longleftarrow i + 1$

Step 11: If $y_0 y_2 > 0$, assign $x_0 = x_2$, otherwise $x_1 = x_2$

Step 12: Print "Solution converges to a root"

Step 13: Print i, $x_2$, $y_2$; goto step 16

Step 14: Print "Initial values unsuitable"

Step 15: Write $x_0, x_1, y_0, y_1$

Step 16: Stop.

(2) Regula – Flasi method

Step 1: Read initial values say $x_0$ and $x_1$

Step 2: Read the prescribed error e and the maximum number of iterations say n

Step 3: $f_0 \longleftarrow f(x_0)$

Step 4: $f_1 \longleftarrow f(x_1)$

Step 5: for $i = 1$ to n

Step 6: $x_2 \longleftarrow (x_0 f_1 - x_1 f_0)/(f_1 - f_0)$

Step 7: $f_2 \longleftarrow f(x_2)$

Step 8: When $|f_2| <= e$ then print the message "Solution is convergent and print $x_2, f_2$ follow step 13; otherwise follow steps 9 to 12

Step 9: When $f_2 f_0 < 0$, assign $x_1 = x_2$ and $f_1 = f_2$ otherwise assign $x_0 = x_2$ and $f_0 = f_2$

Step 10: Next i

Step 11: Print "Does not converge in n iterations"

Step 12: Print $x_2, f_2$

Step 13: Stop

(3) Newton – Raphson method

Step 1: Assign an initial value to x (say $x_0$)

Step 2: Evaluate f(x), $f'(x_0)$ and $f(x_0)$

Step 3: Find the improved estimate of $x_0$

$$x_1 = \frac{x_0 - f(x_0)}{f'(x_0)}$$



Step 4: Check for accuracy of the latest estimate. It may be done by comparing the difference between the values of the latest estimate and the previous estimate to a predefined value, say E. If $|(x_1 - x_0)| < E$. Follow Step 6; otherwise continue.

Step 5: Replace $x_0$ by $x_1$ and repeat steps 3 and 4

Step 6: Stop

(4) Gauss-Elimination method

Step 1: Read n

Step 2: Read $a_{ij}$, $i = 1$ to n and $j = 1$ n+1

Step 3: The equations are arranged such that $a_{11} \neq 0$

Step 4: Eliminate $x_1$ from all except the first equation. For this, normalise the first equation by dividing in by $a_{11}$.

Step 5: Subtract from the second equation 21 times the normalised first equation. Similarly continue the procedure for other equations till the nth equation is finished.

Step 6: Eliminate $x_2$ from the third to the last equation in the new set. Assume that $a_{12} \neq 0$. If $c_{22} = 0$, reorder the equations. Follow the subtraction procedure as stated in Step 5 for $x_2$. Continue this process till the last equation contains only one unknown say $x_n$.

Step 7: Obtain solution by back substitution. This back substitution can be continued till we get the solution for $x_1$.

Step 8: Print the results

Step 9: Stop

(5) Gauss-Jordan method

Step 1: Read n

Step 2: Read the coefficients of the equations say $a_{ij}$, $i = 1$ to n and $j = 1, n+1$

Step 3: Check for the pivotal values, and if they are zero, reorder the equations or rows.

Step 4: Perform normalisation of rows.

Step 5: Eliminate the unknowns (i.e., in kth column)

Step 6: Print the results.

(6) Gauss – Seidel Iterative method

Step 1: Read n

Step 2: Read the coefficients of the n equations say $a_{ij}$ ($i = 1$ to n, $j = 1$ to n)

Step 3: Read initial values of $x_i$, $i = 1$ to n

Step 4: Read $b_i$ ($i = 1$ to n)

Step 5: Check the values of all and if they are zero, print a message "$a_{ii}$ is zero" and follow Step 10, otherwise follow Step 6

Step 6: Start the iterations, to calculate

$$x_i = \frac{b_i - S}{a_{ii}}, \quad i = 1 \text{ to } n$$

where $S = \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} \times j$

Step 7: For each iteration, the absolute difference between the successive values of $x_i$ are computed and the largest difference is stored in E. If E is less than the desired value, follow step 9, otherwise follow step 8.

Step 8: When E is large, repeat the iterations. If the process does not converge after 100 iterations, then follow step 10.

Step 9: Print the result.

Step 10: Stop.

(7) Newton's – Forward formula

Step 1: Read n

Step 2: Read $x_i$ and $y_i$, $i = 1$ to n

Step 3: Read the value of x (say a)

# 4. ALGORITHMS

## 4.0 Definition and origin of an algorithm

The term algorithm may be defined as sequence of instructions designed in such a way that if the instructions are executed in the specified sequence the desired results can be obtained. Further, the instructions should be unambiguous and the result should be obtained after a finite number of executed steps i.e., an algorithm must terminate and should not repeat one or more instructions infinitely. In other words, the algorithm represents the logic of the processing to be performed. The word algorithm originates from the word algorism, which means the process of doing arithmetic with Arabic numerals. Later, the word algorism combined with the word arithmetic to become algorithm.

## 4.1 Development of an algorithm

We first construct an algorithm that gives a very general manner in which computer could produce the solution to a given problem. Such an algorithm is known as general algorithm. In addition, we add details in the general algorithm, in a step-by-step manner, so that the algorithm becomes more detailed, is called refinements of the general algorithm.

The following example shows how an algorithm is written for a given task.

Consider the problem of calculating the average marks of a student in three different subjects.

An algorithm for this task involves the following steps:

Step 1: Read a set of three marks.

Step 2: Find the average by summing them and dividing by three

Step 3: Stop

Successive refinements of this general algorithm add details until complete algorithm is achieved as per the user specifications/requirements. For example, the set of three marks may be obtained from an input device and placed in the variables SUB 1, SUB 2 and SUB 3. Compute the sum and place in the variable TOTMARK. This requires a refinement of step 1 and 2 which is done as follows:

Step 1.1: Obtain three marks through an input device and place them in the variables SUB 1, SUB 2 and SUB 3.

Step 2.1: Compute the total marks and place it in a variable TOTMARK

Step 2.2: Compute the average mark dividing TOTMARK by 3.

It is important to note that hierarchical number system has been used to indicate that these are refinement of steps 1 and 2 of the general algorithm.

If you decide to calculate average marks for a class of 100 students, then the algorithm needs further refinement. The refined algorithm defines a new variable COUNTER which ensures that all the students are taken care for processing.

Step 1: Initialise the counter, say COUNT = 1

Step 2: Read the student register number and his marks from an input device and place them in the variables REGNO, SUB 1, SUB 2 and SUB 3 respectively.

Step 3: Compute the average marks for each student by summing of the corresponding marks and dividing by 3.

Step 4: Display the register number and average marks of each student.

Step 5: Start Increment COUNT BY 1.

Step 6: If COUNT < = 100, repeat steps 1 through 6 otherwise go to

Step 7: Stop

- Solve $\left\{x^4D^4+6x^3D^3+9x^2D^2+3xD+1\right\}y=(1+\log x)^2$,

  Where $D=\frac{d}{dx}$. **(15)**

- Solve $(D^4+D^2+1)y=ax^2+be^{-x}\sin 2x$, where

  $D\equiv\frac{d}{dx}$ **(15)**

- Solve $\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x\sec y$. *(8)*

- Solve and find the singular solution of

  $x^3p^2+x^2py+a^3=0$ *(8)*

- Solve: $x^2y\frac{d^2y}{dx^2}+\left(x\frac{dy}{dx}-y\right)^2=0$ *(10)*

- Solve $\frac{d^4y}{dx^4}+2\frac{d^2y}{dx^2}+y=x^2\cos x$. *(10)*

- Solve $x=y\frac{dy}{dx}-\left(\frac{dy}{dx}\right)^2$ *(1*

- Solve $x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=(1$ *(1*

## 2007

- Find the orthogonal trajectories of the family of the curves $\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$, $\lambda$ being a parameter. (10)
- Show that $e^{2x}$ and $e^{3x}$ are linearly independent solutions of $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$. Find the general solution when $y(0)=0$ and $\left.\frac{dy}{dx}\right]_0=1$ (10)
- Find the family of curves whose tangents form an angle $\pi/4$ with the hyperbola $xy=c$. (10)
- Apply the method of variation of parametes to solve $(D^2+a^2)y=\operatorname{cosec} ax$ (10)
- Find the general solution of $(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+3y=0$ solution of it. (10)

## 2008

- Show that the functions $y_1(x)=x^2$ and $y_2(x)=x^2\log_e x$ are linearly independent obtain the differential equation that has $y_1(x)$ and $y_2(x)$ as the independent solutions. (10)
- Solve the following ordinary differential equation of the second degree $y\left(\frac{dy}{dx}\right)^2+(2x-3)\frac{dy}{dx}-y=0$ (10)
- Reduce the equation $\left(x\frac{dy}{dx}-y\right)\left(x\frac{dy}{dx}\right)=2\frac{dy}{dx}$ to clairaut's form and obtain there by the singular integral of the above equation. (10)
- Solve $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\log_e(1+x)$ (10)
- Find the general solution of the equation $\frac{d^2y}{dx^2}-\cot x\frac{dy}{dx}-(1-\cot x)y=e^x\sin x$. (10)

## 2009

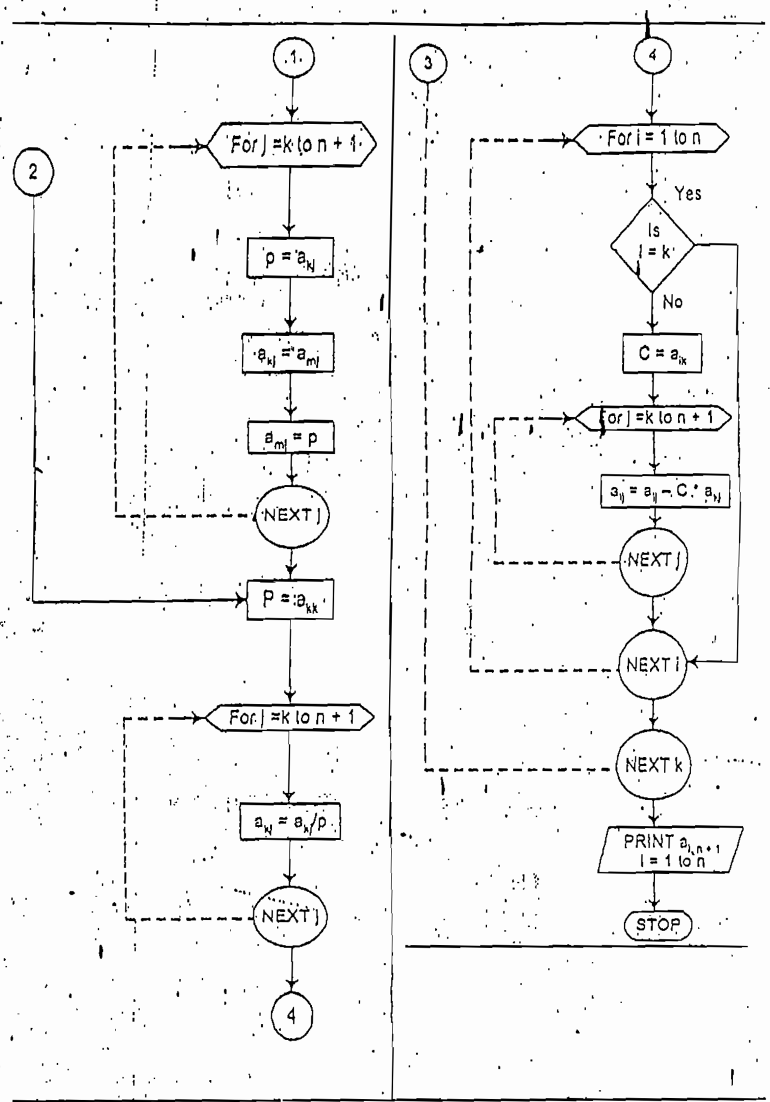
- Solve $\sec^2 y\frac{dy}{dx}+2x\tan y=x^3$ (10)
- Find the 2nd order ODE for which $e^x$ and $x^2e^x$ are solutions. (10)
- Solve $(y^3-2yx^2)dx+(2xy^2-x^3)dy=0$. (10)
- Solve $\left(\frac{dy}{dx}\right)^2-2\frac{dy}{dx}\cosh x+1=0$ (8)
- Solve $\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}+3\frac{dy}{dx}+y=x^2e^{-x}$ (10)
- Show that $e^{x^2}$ is a solution of $\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(4x^2-2)y=0$. *Marks:* 12
  Find a second independent solution.

## 2010

- Show that $\cos(x+y)$ is an integrating factor of $ydx+[y+\tan(x+y)]dy=0$. Hence solve it (8)
- Solve $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=xe^x\sin x$ (8)
- Solve the following differential equation $\frac{dy}{dx}=\sin(x-y+6)$ (8)
- Find the general solution of $\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+(x^2+1)y=0$ (12)
- Solve $\left(\frac{d}{dx}-1\right)^2\left(\frac{d^2}{dx^2}+1\right)^2 y=x+e^x$ (10)
- Solve by the method of variation of parameters the following eqaution $(x^2-1)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=(x^2-1)^2$ (10)

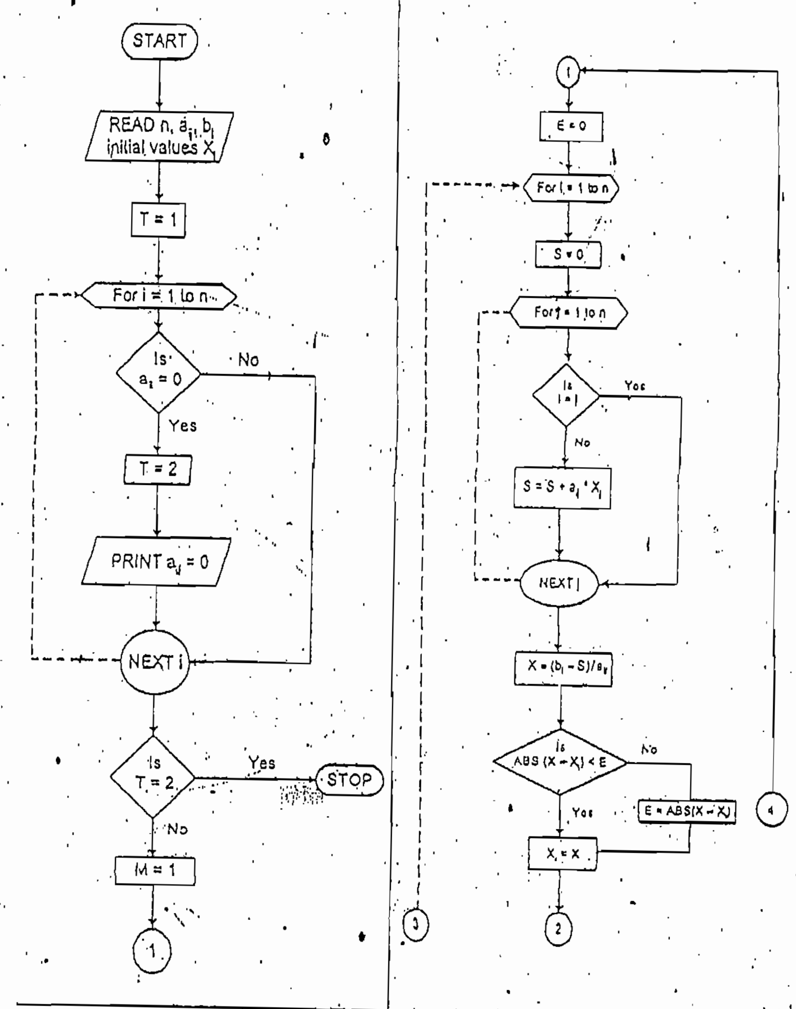
## 2011

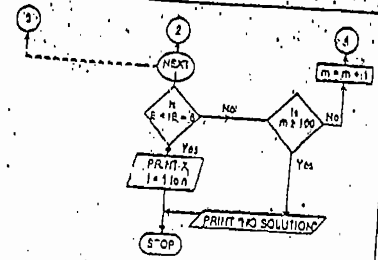
- Find the family of curves whose tangents form an angle $\pi/4$ With hyperbolas $xy=c$. (10)
- Solve $\frac{d^2y}{dx^2}-2\tan x\frac{dy}{dx}+5y=\sec x\, e^x$. (10)
- Solve $p^2+2py\cot x=y^2$ Where $p=\frac{dy}{dx}$. (10)



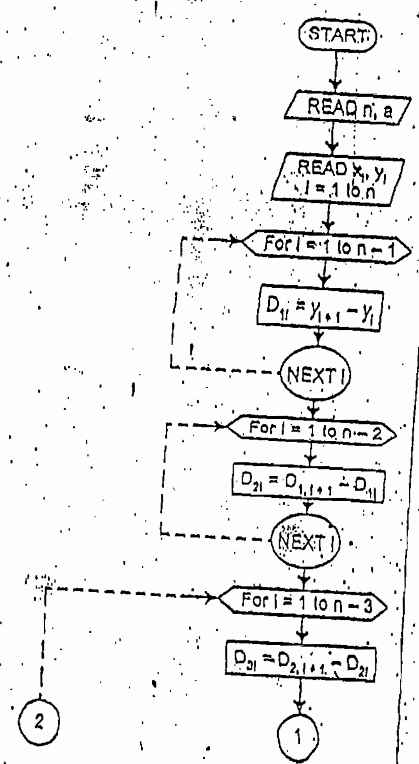


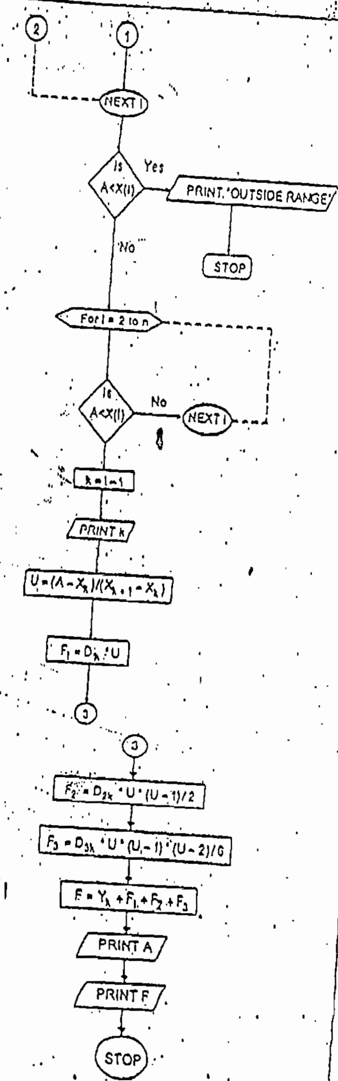

(6) Flow chart for Gauss-Seidel (iterative) method





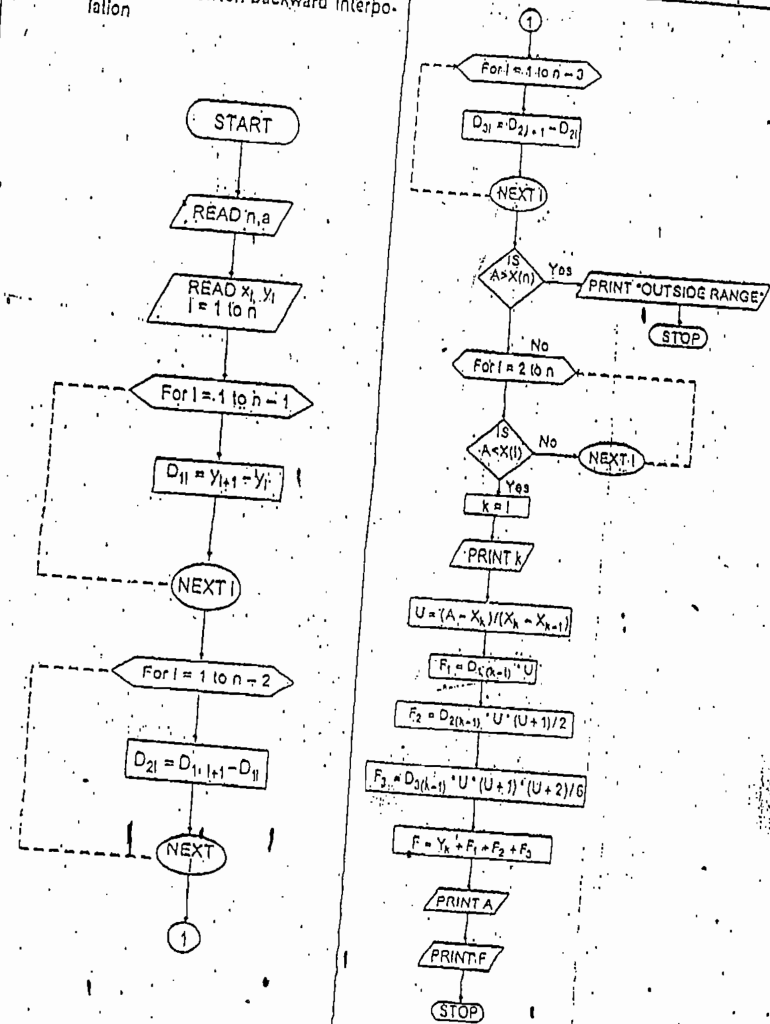

(7) Flow Chart for Newton-forward Interpolation






(8) Flow chart Newton-backward Interpolation



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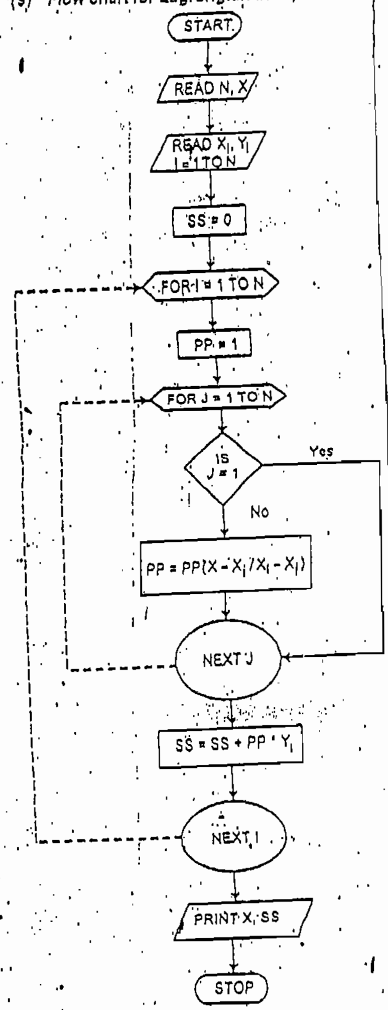
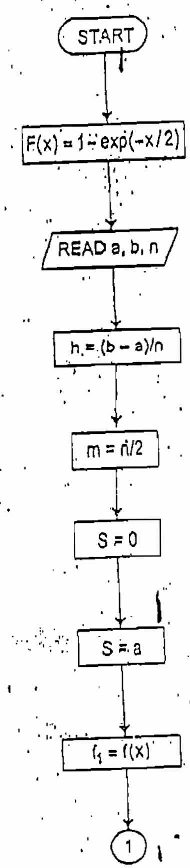
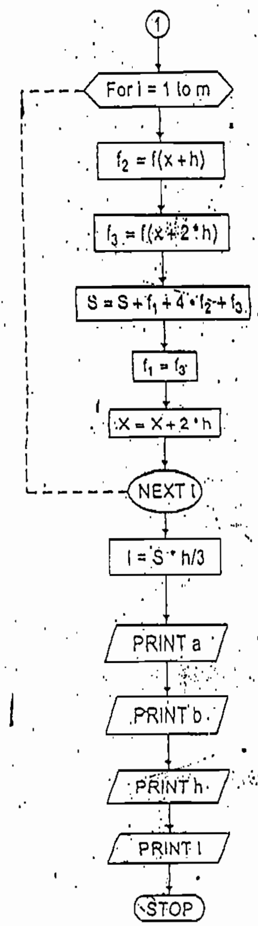
Mob: 09999197625

(9) Flow chart for Lagrangian Interpolation

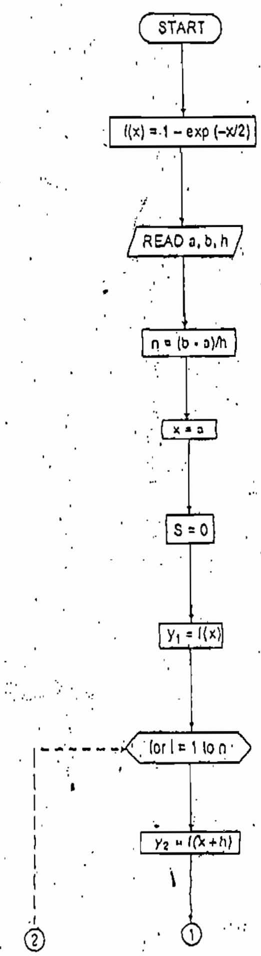


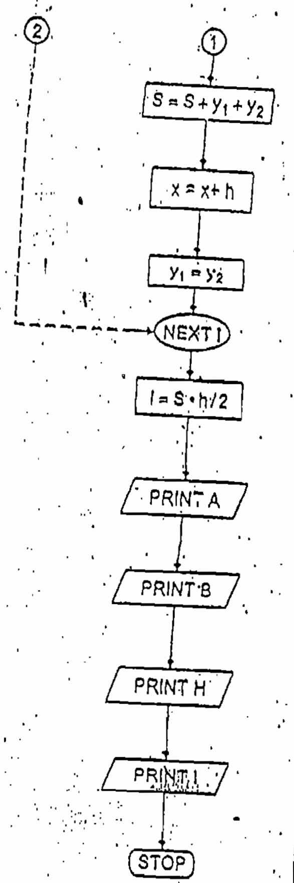

(10) Flow chart for Simpson's one-third rule



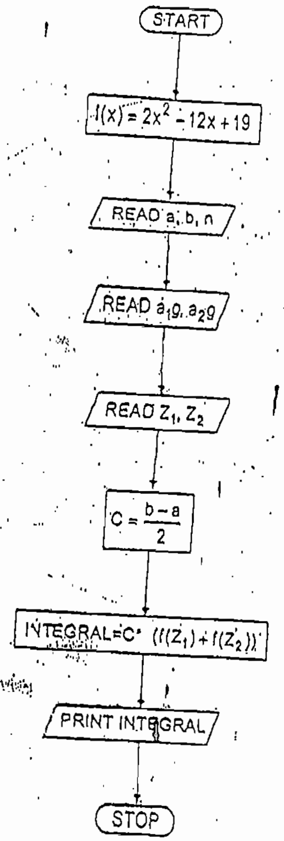

(11) Flow chart for Trapezoidal rule



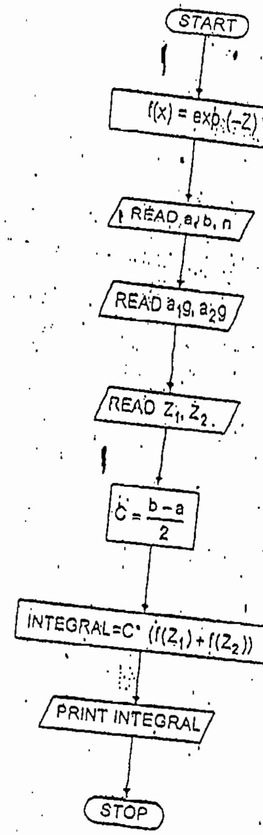



(12) Flow chart for Gaussian Quadrature formula

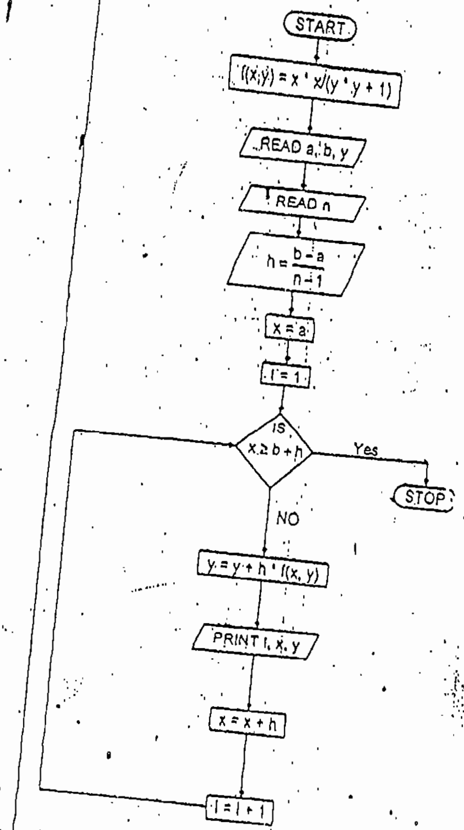
(a)




(b)




(13) Flow chart for Euler's method



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(14). Flow chart for 4th order Runge-Kutta method